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The homology of homotopy inverse limits[1]

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Abstract

The homology of a homotopy inverse limit can be studied by a spectral sequence which has as the E_2 term the derived functor of limit in the category of coalgebras. These derived functors can be computed using the theory of Dieudonné modules if one has a diagram of connected abelian Hopf algebras.

0. Introduction

One of the standard problems in homotopy theory is to calculate the homology of a given type of inverse limit. For example, one might want to know the homology of the inverse limit of a tower of fibrations, or of the pull-back of a fibration, or of the homotopy fixed point set of a group action, or even of an infinite product of spaces. This paper presents a systematic method for dealing with this problem and works out a series of examples.

It simplifies the foundational questions present when dealing with inverse limits to work with simplicial sets rather than topological spaces. So this paper is written simplicially: that is, a space is a simplicial set. As usual this does not affect the homotopy theory. Homology is with $\mathbb{F}_p$ coefficients.

Here are some simple examples of the type of result we obtain. An abelian Hopf algebra is one for which both diagonal and multiplication are commutative.

Theorem A. *Let $\{X_\alpha\}$ be an arbitrary set of connected nilpotent spaces and suppose for all α, H_*X_α is an abelian Hopf algebra. Then there is a natural isomorphism*

$$H_*\Big(\prod_\alpha X_\alpha\Big) \cong \prod_\alpha H_*X_\alpha.$$

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This is a companion to a result of Bousfield's [3, 4.4] where slightly different hypotheses yield a slightly different result. The product $\prod_\alpha H_* X_\alpha$ is in the category of graded connected coalgebras. If the set is finite it is simply the graded tensor product. If the set is infinite something more sophisticated is required. See Section 1 for limits of coalgebras.

Theorem B. *Consider a tower of fibrations over the natural numbers*

$$X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$$

so that for all i, X_i is connected and $H_* X_{i+1} \to H_* X_i$ is a morphism of abelian Hopf algebras. Suppose each X_i is simple and $\lim^1 \pi_1 X_i = 0$. Then under either of the following conditions, there is an isomorphism

$$H_*(\lim X_i) \cong \lim_i H_* X_i:$$

(1) *for all i, $H_* X_i$ is of finite type;*
(2) *the tower $H_* X_1 \leftarrow H_* X_2 \leftarrow \cdots$ is Mittag–Leffler.*

These are two of the usual conditions for vanishing of $\lim^1$ for abelian groups. Again $\lim_i H_* X_i$ must be calculated in the category of coalgebras. Under various conditions, however, this will be isomorphic to the inverse limit as vector spaces. One common example of this is if the tower $\{H_* X_i\}_i$ is pro-isomorphic, as a tower of Hopf algebras, to a constant tower. This is a stronger condition than Mittag–Leffler.

Behind these results is a technique which I will now explain. In general, given a small category I and an I-diagram X of spaces there is a map of graded coalgebras

$$H_*\left(\operatorname{holim}_I X\right) \to \lim_I H_* X,$$

where holim is the Bousfield–Kan homotopy inverse limit and the limit on the right is in the category of coalgebras. There is no reason to think that is either injective or surjective; however, it is the edge homomorphism of a spectral sequence. This spectral sequence is a variant of one due to Anderson [1] and studied by Bousfield in [3]. The major advantage of our variant is that the E_2 term can be identified and computed by non-abelian homological algebra.

Let $\mathscr{CA}_+$ be the category of graded connected coalgebras over $\mathbb{F}_p$ and $\mathscr{CA}_+^I$ the category of I-diagrams in $\mathscr{CA}_+$. The point of this paper is that the functor

$$\lim_I : \mathscr{CA}_+^I \to \mathscr{CA}_+$$

has right derived functors $R^s_{\mathscr{CA}} \lim_I$, $s \geq 0$, so that $R^0_{\mathscr{CA}} \lim_I \cong \lim_I$ and the bigraded vector space $R^*_{\mathscr{CA}} \lim_I C$ is a bigraded, cocommutative coalgebra. Then we have

Theorem C. *Let $X : I \to \mathbf{S}$ be an I-diagram of connected spaces. Then there is a natural second quadrant homology spectral sequence:*

$$R^*_{\mathscr{CA}} \lim_I H_* X \Rightarrow H_* \operatorname{holim}_I X_p.$$

Here $X_p: I \to \mathrm{S}$ is the I-diagram obtained by applying the Bousfield–Kan p-completion functor to X. Because this is a second quadrant spectral sequence, convergence is a problem. For this we appeal to [3] or [13]. This accounts for some of the hypotheses of Theorems A and B. Beyond this there is the problem of computing $R^s_{\mathscr{C}\mathscr{A}} \lim_I H_* X$. For example Theorems A and B are based on the assertion that $R^s_{\mathscr{C}\mathscr{A}} \lim_I H_* X = 0$ for $s > 0$ under the listed hypotheses. It is here that we use the assumption that $H_* X : I \to \mathscr{C}\mathscr{A}_+$ is actually a diagram of abelian Hopf algebras. The category of abelian Hopf algebras is equivalent to a category of modules called Dieudonné modules and one can pass from the ordinary derived functors of $\lim_I$ for modules to $R^*_{\mathscr{C}\mathscr{A}} \lim_I$ at least in some cases. We give examples, at least when $\lim^s_I = 0$ for $s > 1$ as modules. Even in this case, $R^s_{\mathscr{C}\mathscr{A}} \lim_I$ may not be zero for any $s \geq 0$. Using these computations we make some calculations of the spectral sequence in cases where $R^*_{\mathscr{C}\mathscr{A}} \lim_I H_* X$ is not concentrated in degree zero. Included is an example of homotopy fixed points. Finally, there is the problem of relating $H_* \operatorname{holim} X$ to $H_* \operatorname{holim} X_p$. Again there are techniques in [3] that apply.

Despite the title of this paper and the general thrust of this introduction, this is primarily a work in non-abelian homological algebra – five of the six sections are devoted to the definitions and calculations of $R^*_{\mathscr{C}\mathscr{A}} \lim_I$. Only the sixth and last section contains homotopy theoretic applications. Sections 1 and 2 are devoted to the definition and homotopical algebra foundations of $R^*_{\mathscr{C}\mathscr{A}} \lim_I$ – these derived functors are, in fact, the cohomotopy groups of a homotopy inverse limit of cosimplicial coalgebras, although this language is avoided until Section 2. Section 3 is on Dieudonné modules, Section 4 gives some calculations, and Section 5 contains the proofs of some technical lemmas.

The paper began as a meditation on Example 4.4 of [3], spurred on by a conversation with John Hunton. In fact, the proof of Theorem A given here may be regarded as a wildly expanded version of Bousfield's argument, and, in general, this paper owes a great debt to [3]. Finally, a conversation with Brooke Shipley on bicosimplicial spaces was useful for widening my scope. Several results for [13] make this paper go much more smoothly.

1. Derived functors of limits in the category of coalgebras

Let $\mathscr{C}\mathscr{A}$ be the category of graded cocommutative coalgebras over a field k. Later, we will specialize to the case where $k = \mathbb{F}_p$. As always in homotopy theory applications, commutativity is with a sign. Thus if $C \in \mathscr{C}\mathscr{A}$ and $x \in C$ has diagonal

$$\Delta x = \sum y_i \otimes z_i$$

then

$$\sum y_i \otimes z_i = \sum (-1)^{|y_i||z_i|} z_i \otimes y_i,$$

where $|w|$ is the degree of w. Let $\mathscr{CA}_+ \subseteq \mathscr{CA}$ denote the full sub-category of *connected coalgebras*. Thus $\mathscr{C} \in \mathscr{CA}_+$ if and only if $C_0 \cong \mathbb{F}_p$.

A useful fact is:

Lemma 1.1. *Let $C \in \mathscr{CA}$. Then C is isomorphic to the filtered colimit of its finite dimensional sub-coalgebras.*

Proof. This is clear in $C \in \mathscr{CA}_+$. The general case follows from [14, p. 46]. □

Now let I be a small category and $\mathscr{CA}^I$ the category of I-diagram in $\mathscr{CA}$. By definition,

$$\lim_I : \mathscr{CA}^I \to \mathscr{CA}$$

is right adjoint to the constant diagram functor. Such limits exist for all I. To see this, first suppose I is filtered. Then if $C : I \to \mathscr{CA}$ is an I-diagram we can form the vector space limit $\lim_I^k C$. This is not, in general, a coalgebra; however, it is a complete coalgebra in the following sense. For $i \in I$, let

$$E(i) = \ker \{ \lim_I^k C \to C(i) \}.$$

Then $\lim_I^k C$ is a complete topological vector space with respect to the neighborhood base for zero $\{E(i)\}$. Then there is a coproduct

$$\lim_I^k C \to \lim_I^k C \mathbin{\hat{\otimes}} \lim_I^k C,$$

where $\hat{\otimes}$ denotes the completion of the tensor product with respect to $\{E(i) \otimes E(j)\}$. This is because

$$\lim_I^k C \mathbin{\hat{\otimes}} \lim_I^k C \cong \lim_I^k (C \otimes C).$$

A sub-coalgebra $D \subseteq \lim_I^k C$ is a sub-vector space equipped with a coproduct $D \to D \otimes D$ making D a coalgebra such that the evident diagram commutes:

$$\begin{array}{ccc} D & \longrightarrow & D \otimes D \\ \downarrow & & \downarrow \\ \lim^k C & \longrightarrow & \lim^k C \mathbin{\hat{\otimes}} \lim^k C \end{array}$$

Then, for the limit in the category of coalgebras,

$$\lim_I C = \operatorname{colim}_\alpha D_\alpha \tag{1.1a}$$

where D_α runs over all sub-coalgebras over $\lim_I^k C$. Because of Lemma 1.1, we could equally require the D_α to be finite. Next, suppose I is discrete, so that $\lim_I = \prod_I$. In the object set of I is finite, then

$$\prod_I C = \bigotimes_I C.$$

Then, in general,

$$\prod_I C = \lim_J \prod_J C, \tag{1.1b}$$

where lim is in the category $\mathscr{C}\mathscr{A}$ and J runs over the filtered diagram of finite sub-categories of I.

Proposition 1.2. *The category $\mathscr{C}\mathscr{A}$ has all the limits.*

Proof. We now have products, by (1.1b), so to get all limits we need only supply pull-backs. However, the pull-back of a diagram

$$C_1 \to C_{12} \leftarrow C_2$$

is the cotensor product $C_1 \square_{C_{12}} C_2$. □

Because of the colimit in formula (1.1a) $\lim_I$ will not, in general, preserve surjections, even in the case of (infinite) products (1.1b). Thus one will have derived functors. There are several possible definitions of these. After some preliminaries, we will give a definition in Definition 1.3 which, while complicated, arises in homotopy theory applications.

Let $J:\mathscr{C}\mathscr{A}_+ \to n_+k$ be the "coaugmentation coideal" functor from connected coalgebras to positively graded vector spaces. Thus JC is nothing more than the elements of positive degree. The functor J has a right adjoint S with $SW = \oplus_{n\geq 0} S^n W$ where $S^0 W = k$ and

$$S^n W = (\underbrace{W \otimes \cdots \otimes W}_{n})^{\Sigma_n},$$

where Σ_n acts by permuting factors, up to signs required by the grading. If W is of finite type, $(SW)^*$ is the free graded commutative algebra on W^* and, hence, if $\mathrm{char}(k) \neq 2$, is a tensor product of polynomial and exterior algebras.

Let $\bar{S} = S \circ J:\mathscr{C}\mathscr{A}_+ \to \mathscr{C}\mathscr{A}_+$ be the composite functor. It is the underlying functor of a triple on $\mathscr{C}\mathscr{A}_+$. Thus if $C \in \mathscr{C}\mathscr{A}_+$, one has a canonical cosimplical resolution

$$\eta: C \to \bar{S}^\bullet C. \tag{1.2}$$

In particular, $\pi^* \bar{S}^\bullet C \cong C$ with isomorphism induced by η. This can be seen by applying J to (1.2) and noting that the resulting cosimplicial vector space has a natural contraction.

This resolution can be used to define derived functors. For example, if P is the primitive element functor

$$R^s PC = \pi^s P \bar{S}^\bullet C. \tag{1.3}$$

These derived functors were one of the main topics of [4]. We also use this resolution to define derived functors of limit. But we need another preliminary.

Fix a small category I. The classifying space of the category I is the simplicial set BI with 0-simplices the objects of I and n-simplices, $n \geq 1$, strings of arrows in I

$$\vec{i} = i_n \to i_{n-1} \to \cdots \to i_1 \to i_0$$

with face and degeneracy operators given by

$$\begin{aligned}
d_0 \vec{i} &= i_n \to i_{n-1} \to \cdots \to i_1, \\
d_j \vec{i} &= i_n \to \cdots \to i_{j+1} \to i_{j-1} \to \cdots \to i_0, \\
d_n \vec{i} &= i_{n-1} \to \cdots \to i_1 \to i_0, \\
s_j \vec{i} &= i_n \to \cdots \to \cdots i_j \xrightarrow{1} i_j \to \cdots \to i_0.
\end{aligned}$$

Note that for d_j, $i_{j+1} \to i_{j-1}$ is the composition. Now let $\mathscr{C}$ be a category with products and $X: I \to \mathscr{C}$ an I-diagram. Then we get a cosimplicial object $\Pi^\bullet X$ in $\mathscr{C}$ with

$$\prod^n X = \prod_{\vec{i} \in BI_n} X(i_0) \tag{1.4}$$

with coface and codegeneracy maps induced by those in BI, with the evident twist in d^0: the composite

$$\prod_{\vec{i} \in BI_{n-1}} X(i_0) \xrightarrow{d^0} \prod_{\vec{j} \in BI_n} X(j_0) \xrightarrow{\pi_{\vec{j}}} X(j_0)$$

fits into a commutative diagram

$$\begin{array}{ccc}
\Pi_{\vec{i}}\, X(i_0) & \xrightarrow{d^0} & \Pi_{\vec{j}}\, X(j_0) \\
\downarrow{\scriptstyle \pi_{d_0 \vec{j}}} & & \downarrow{\scriptstyle \pi_{\vec{j}}} \\
X(j_1) & \longrightarrow & X(j_0)
\end{array}$$

Note that if $A: I \to \mathbf{ab}$ is a diagram of abelian groups, then

$$\lim_I{}^s A \cong \pi^s \Pi^\bullet A. \tag{1.5}$$

See [5, p. 305].

Now let $C: I \to \mathscr{C}\mathscr{A}_+$ be an I-diagram of connected coalgebras. Then for each $i \in I$, one can form the augmented cosimplicial coalgebra

$$\eta: C(i) \to \bar{S}^\bullet C(i)$$

and the naturality of this construction yields an I-diagram of augmented cosimplicial coalgebras. Hence one obtains a bicosimplicial coalgebra $\Pi^\bullet \bar{S}^\bullet C$ which has a diagonal cosimplicial coalgebra diag $\Pi^\bullet \bar{S}^\bullet C$.

Definition 1.3. Let $C: I \to \mathscr{CA}_+$ be an I-diagram of coalgebras. Then the derived functors of $\lim_I$ are defined by

$$R^s_{\mathscr{CA}} \lim_I C = \pi^s \operatorname{diag}(\Pi^\bullet \bar{S}^\bullet C).$$

Remarks 1.4. (1) The augmentation $\eta: C \to \bar{S}^\bullet C$ induces a map

$$\Pi^\bullet C \to \operatorname{diag} \Pi^\bullet \bar{S}^\bullet C$$

which may or may not be an equivalence. See Example 1.5 below and Corollary 4.3.

(2) Since $R^*_{\mathscr{CA}} \lim_I C$ is the cohomotopy of a cosimplicial coalgebra, it is a bigraded cocommutative coalgebra. The switch sign depends on the total degree.

(3) $R^0_{\mathscr{CA}} \lim_I C \cong \lim_I C$ since π^0 diag $\Pi^\bullet \bar{S}^\bullet C$ is an equalizer and limits commute.

The remainder of this section is devoted to two examples.

Example 1.5. If I has a finite object set and $C: I \to \mathscr{CA}_+$ is an I-diagram – so C is a finite diagram – then

$$\eta: \Pi^\bullet C \to \operatorname{diag} \Pi^\bullet \bar{S}^\bullet C$$

is a π^* isomorphism. To see this, filter the bicosimplicial coalgebra

$$\Pi^\bullet \bar{S}^\bullet C = \{\Pi^n \bar{S}^m C\}$$

by degree in n to get a spectral sequence

$$E_1^{n,m} = \pi^m(\pi^n \bar{S}^\bullet C) \Rightarrow \pi^{n+m} \operatorname{diag} \Pi^\bullet \bar{S}^\bullet C.$$

Since I is finite, BI_n is finite, so $\Pi^n \bar{S}^\bullet C = \otimes_{i \in BI_n} \bar{S}^\bullet C(i_0)$ and the Eilenberg–Zilber Theorem says

$$E_1^{n,m} = \begin{cases} 0, & m > 0, \\ \Pi^n C, & m = 0. \end{cases}$$

The claim follows. Thus one has

$$\pi^*(\Pi^\bullet C) \cong R^*_{\mathscr{CA}} \lim_I C.$$

If I is finite and discrete, one has $R^n_{\mathscr{CA}} \lim_I C = 0$ for $n > 0$ and $R^0_{\mathscr{CA}} \lim_I C \cong \Pi_I C \cong \otimes_I C$. If I is the category with diagram $1 \to 12 \leftarrow 2$, so and I-diagram of the form

$$C_1 \to C_{12} \leftarrow C_2$$

one has

$$R^*_{\mathscr{C}\mathscr{A}} \lim_I C \cong \pi^*(\Pi^\bullet C) \cong \mathrm{Cotor}^*_{C_{12}}(C_1, C_2)$$

as we would hope.

Example 1.6. Let I be discrete, but not necessarily finite. Then I claim the other augmentation

$$\prod_I \bar{S}^\bullet C \to \Pi^\bullet \bar{S}^\bullet C$$

induces a π^* isomorphism

$$\prod_I \bar{S}^\bullet C \to \mathrm{diag}\, \Pi^\bullet \bar{S}^\bullet C$$

so that

$$\pi^*\left(\prod_I \bar{S}^\bullet C\right) \cong R^*_{\mathscr{C}\mathscr{A}}\left(\prod_I\right) C.$$

To see this, filter $\Pi^\bullet \bar{S}^\bullet C = \{\Pi^n \bar{S}^m C\}$ by degree in m, to get a spectral sequence

$$E_1^{n,m} = \pi^n \Pi^\bullet \bar{S}^m C \Rightarrow \pi^{n+m} \mathrm{diag}\, \Pi^\bullet \bar{S}^\bullet C.$$

For fixed m, $\bar{S}^m C = (\bar{S} \circ \cdots \circ \bar{S})C$, with the composition taken $m+1$ times. Then

$$\Pi^\bullet \bar{S}^m C \cong S(\Pi^\bullet J \circ \bar{S}^{m-1} C)$$

since S is a right adjoint and $\Pi^\bullet$ on the inside is in the category of vector spaces. Then

$$\pi^n \Pi^\bullet J \circ \bar{S}^{m-1} C \cong \lim_I{}^n (J \circ \bar{S}^{m-1} C)$$

$$\cong \begin{cases} 0, & n > 0, \\ \prod_I J \circ \bar{S}^{m-1} C, & n = 0, \end{cases}$$

since these derived functors are in vector spaces. Thus $\Pi^\bullet J \circ \bar{S}^{m-1} C$ has a cosimplicial contraction and

$$E_1^{n,m} = \begin{cases} 0, & n > 0, \\ \prod_I \bar{S}^m C, & n = 0. \end{cases}$$

where this product is in $\mathscr{C}\mathscr{A}_+$. The result follows.

2. The homotopical foundations of $R^*_{\mathscr{C}\mathscr{A}} \lim_I$

The point of this section is to demonstrate that the derived functors of limit in the category $\mathscr{C}\mathscr{A}$ can be given a homotopical foundation flexible enough to be useful in computations. Afterward we supply some examples.

Let $c\mathscr{C}\mathscr{A}_+$ be the category of cosimplicial coalgebras. We will show that $R^*_{\mathscr{C}\mathscr{A}}\lim_I$ are the cohomotopy groups of a homotopy inverse limit in $c\mathscr{C}\mathscr{A}_+$.

To begin with $c\mathscr{C}\mathscr{A}_+$ is a simplicial model category in the sense of Quillen [11, II.2]. This is a result of [7]; see also [2], although the latter reference is not explicit about the simplicial structure. To spell out the details, first note that $c\mathscr{C}\mathscr{A}_+$ is a simplicial category by the results of Quillen. Indeed if $K \in S$ is a simplicial set and $C^\bullet \in c\mathscr{C}\mathscr{A}_+$ then $\hom(K, C^\bullet) \in c\mathscr{C}\mathscr{A}_+$ is the cosimplicial coalgebra with

$$\hom(K, C^\bullet)^n = \prod_{k \in K_n} C^n \tag{2.1}$$

with the induced face and codegeneracy operators. There is also an object $C^\bullet \otimes K$ uniquely determined by the natural isomorphism

$$\mathrm{Hom}_{c\mathscr{C}\mathscr{A}_+}(C^\bullet \otimes K, D^\bullet) \cong \mathrm{Hom}_{c\mathscr{C}\mathscr{A}_+}(C^\bullet, \hom(K, D^\bullet)). \tag{2.2}$$

Finally, there is a mapping space functor on $c\mathscr{C}\mathscr{A}_+$ with n-simplices given by

$$\mathrm{map}_{c\mathscr{C}\mathscr{A}_+}(C^\bullet, D^\bullet) = \mathrm{Hom}_{c\mathscr{C}\mathscr{A}_+}(C^\bullet, \hom(\Delta^n, D^\bullet)). \tag{2.3}$$

Definition 2.1. A morphism $f: C^\bullet \to D^\bullet$ is

(1) a weak equivalence if $\pi^* f$ is an isomorphism;

(2) a cofibration if the induced map $Nf: NC^\bullet \to ND^\bullet$ of normalized cochain complexes is an injection in positive degrees;

(3) a fibration if it has the right lifting property with respect to all maps which are at once cofibrations and weak equivalences.

Then one has:

Theorem 2.2. (Kuczmarski [7]). *With these definitions, $c\mathscr{C}\mathscr{A}_+$ becomes a simplicial model category.*

2.3. Part of the proof is to describe fibrations. A morphism $f: C^\bullet \to D^\bullet$ *is almost-free* if for all $n \geq 0$ there are vector spaces W^n and maps $\sigma^i: W^n \to W^{n-1}$, $0 \leq i \leq n-1$, so that

(1) there are isomorphisms $C^n \cong D^n \otimes S(W^n)$ that fit into commutative diagrams

$$\begin{array}{ccc} C^n & \xrightarrow{\cong} & D^n \otimes S(W^n) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ D^n & \xrightarrow{=} & D^n \end{array}$$

(2) the following diagrams commute

$$\begin{array}{ccc} C^n & \xrightarrow{\quad s^i \quad} & C^{n-1} \\ \downarrow \cong & & \downarrow \cong \\ D^n \otimes S(W^n) & \xrightarrow{\quad s^i \otimes S\sigma^i \quad} & D^{n-1} \otimes S(W^{n-1}) \end{array}$$

Proposition 2.4. *Almost-free maps are fibrations in $c\mathscr{C}\mathscr{A}_+$ and any fibration is a retract of an almost-free map.*

If the unique map $\varepsilon : C^\bullet \to k$ is a fibration, $C^\bullet$ will be called fibrant; if ε is almost-free, $C^\bullet$ will be called almost-free. Since any retract of a coalgebra of the form $S(W)$ is of the form $S(W_0)$ (by [4, Proposition 4.2]), one has

Corollary 2.5. *An object is fibrant if and only if it is almost-free.*

Example 2.6. Let $C \in \mathscr{C}\mathscr{A}_+$ be regarded as a constant cosimplicial coalgebra. Then the canonical map $\eta : C \to \bar{S}^\bullet C$ is a cofibration and a weak equivalence, and $\bar{S}^\bullet C$ is fibrant.

Because $c\mathscr{C}\mathscr{A}_+$ is a simplicial model category with all products and coproducts, it has homotopy inverse limits in the style of Bousfield and Kan [5, XI, §4ff]. To recapitulate. Let I be a small category. For $i \in I$, let $I \downarrow i$ be the category of objects over I and let $B(I \downarrow i)$ be its classifying space. Each of these spaces is contractible since $I \downarrow i$ has a terminal object. The assignment

$$i \mapsto B(I \downarrow i)$$

is an I-diagram of spaces. Now let $C^\bullet : I \to c\mathscr{C}\mathscr{A}_+$ be an I-diagram of cosimplicial coalgebras. Then $\operatorname{holim} C^\bullet = \operatorname{holim}_I C^\bullet$ is defined by an equalizer diagram in $c\mathscr{C}\mathscr{A}_+$

$$\operatorname{holim} C^\bullet \to \prod_i \hom(B(I \downarrow i), C^\bullet(i)) \underset{d^0}{\overset{d^1}{\rightrightarrows}} \prod_{i \to i'} \hom(B(I \downarrow i), C^\bullet(i')), \tag{2.4}$$

where d^0 and d^1 are induced, respectively, by

$$\hom(B(I \downarrow i, C^\bullet(i)) \to \hom(B(I \downarrow i), C^\bullet(i'))$$

and

$$\hom(B(I \downarrow i'), C^\bullet(i')) \to \hom(B(I \downarrow i), C^\bullet(i')).$$

However, combining this definition with the description of $\hom(K, C^\bullet)$ given in (2.1) one has

$$\operatorname{holim} C^\bullet \cong \operatorname{diag}(\Pi^\bullet C^\bullet), \tag{2.5}$$

where $\Pi^\bullet$ is the cosimplicial construction of the previous section.

One would like holim $C^\bullet$ to have the following invariance property: if $C^\bullet(i) \to D^\bullet(i)$ is a weak equivalence for all i, then

$$\operatorname{holim} C^\bullet \to \operatorname{holim} D^\bullet$$

should be a weak equivalence. This fails in general; however, one has:

Lemma 2.7. *Let $C^\bullet \to D^\bullet$ be a morphism of I-diagrams in $c\mathscr{C}\mathscr{A}_+$ and suppose $C^\bullet(i) \to D^\bullet(i)$ is a weak equivalence for all $i \in I$. Then if $C^\bullet(i)$ and $D^\bullet(i)$ are fibrant for all i,*

$$\operatorname{holim} C^\bullet \to \operatorname{holim} D^\bullet$$

is a weak equivalence.

Proof. Write the bicosimplicial vector space

$$\Pi^\bullet C^\bullet = \{\Pi^p C^q\}.$$

Filtering by degree in p gives a spectral sequence

$$\begin{aligned} E_1^{p,q} = \pi^q(\Pi^p C^\bullet) &\Rightarrow \pi^{p+q}\operatorname{diag}(\Pi^\bullet C^\bullet) \\ &\cong \pi^{p+q} \operatorname{holim} C^\bullet. \end{aligned}$$

Thus we need only show $\Pi^p C^\bullet \to \Pi^p D^\bullet$ is a weak equivalence for all p. Since every object of $c\mathscr{C}\mathscr{A}_+$ is cofibrant, any weak equivalence between fibrant objects is a homotopy equivalence, which, in $c\mathscr{C}\mathscr{A}_+$ is the same as a cosimplicial homotopy equivalence in the sense of [10]. Thus $\Pi^p C^\bullet \to \Pi^p D^\bullet$ is a cosimplical homotopy equivalence, hence a weak equivalence. □

For this reason one demands, before taking a homotopy inverse limit in $c\mathscr{C}\mathscr{A}_+$, that one has a diagram of fibrant objects. One of the results of [7] is that for all $C^\bullet \in c\mathscr{C}\mathscr{A}_+$, there is a natural cofibration and weak equivalence $C^\bullet \to D^\bullet$ with $D^\bullet$ fibrant. Thus if $C^\bullet : I \to c\mathscr{C}\mathscr{A}_+$ is an I-diagram $D^\bullet : I \to c\mathscr{C}\mathscr{A}_+$ is an I-diagram of fibrant objects and one sets

$$\mathbf{R}\operatorname{holim} C^\bullet = \operatorname{holim} D^\bullet. \qquad (2.6)$$

The boldface **R** means Quillen's total right derived functor.

Corollary 2.8. *If $C_1^\bullet \to C_2^\bullet$* is a morphism of *$I$-diagrams in $c\mathscr{C}\mathscr{A}_+$ and $C_1^\bullet(i) \to C_2^\bullet(i)$ is a weak equivalence for all i, then*

$$\mathbf{R}\operatorname{holim} C_1^\bullet \to \mathbf{R}\operatorname{holim} C_2^\bullet$$

is a weak equivalence.

Proof. Combine Lemma 2.7 with the definition (2.6). □

More is true however.

Lemma 2.9. *Let $C^\bullet : I \to c\mathscr{C}\mathscr{A}_+$ be an I-diagram of cosimplicial coalgebras and let $C^\bullet \to C_1^\bullet$ be a morphism of I diagrams so that*

$$C^\bullet(i) \to C_1^\bullet(i)$$

is a weak equivalence for all i and $C_1^\bullet(i)$ is fibrant for all i. Then there is a weak equivalence

$$\mathbf{R}\,\mathrm{holim}\, C^\bullet \simeq \mathrm{holim}\, C_1^\bullet.$$

Proof. Consider the diagram

$$\begin{array}{ccc} C^\bullet & \longrightarrow & D^\bullet \\ \downarrow & & \downarrow \\ C_1^\bullet & \longrightarrow & D_1^\bullet \end{array}$$

where $D^\bullet$ and $D_1^\bullet$ are the natural fibrant replacements indicated above. Then Lemma 2.7 guarantees weak equivalences

$$\mathrm{holim}\, C_1^\bullet \to \mathrm{holim}\, D_1^\bullet \leftarrow \mathrm{holim}\, D^\bullet. \qquad \square$$

Example 2.10. Let $C : I \to c\mathscr{C}\mathscr{A}_+$ be an I-diagram of coalgebras. We may regard each $C(i)$ as a constant cosimplicial coalgebra. Then, by Example 2.6, $\eta : C \to \bar{S}^\bullet C$ is a morphism of I-diagrams in $c\mathscr{C}\mathscr{A}_+$ so that $C(i) \to \bar{S}^\bullet C(i)$ is a weak equivalence with $\bar{S}^\bullet C(i)$ fibrant. Hence

$$\mathbf{R}\,\mathrm{holim}\, C = \mathrm{holim}\, \bar{S}^\bullet C \cong \mathrm{diag}(\Pi^\bullet \bar{S}^\bullet C),$$

$$\pi^* \mathbf{R}\,\mathrm{holim}\, C \cong R^*_{\mathscr{C}\mathscr{A}} \lim_I C.$$

As a first example of the kind of flexibility we've acquired, I supply:

Example 2.11. Let $C_1, C_2 : I \to \mathscr{C}\mathscr{A}_+$ be two I-diagrams. Then there is a natural isomorphism of bigraded coalgebras

$$R^*_{\mathscr{C}\mathscr{A}} \lim_I (C_1 \otimes C_2) \cong R^*_{\mathscr{C}\mathscr{A}} \lim_I C_1 \otimes R^*_{\mathscr{C}\mathscr{A}} \lim_I C_2.$$

As a result of this, if $H : I \to \mathscr{C}\mathscr{A}_+$ is actually a diagram of Hopf algebras. $R^*_{\mathscr{C}\mathscr{A}} \lim_I H$ is a bigraded Hopf algebra. If H is commutative, so is $R^*_{\mathscr{C}\mathscr{A}} \lim_I H$.

To see the isomorphism, I claim there is a weak equivalence

$$\mathbf{R}\,\mathrm{holim}(C_1 \otimes C_2) \simeq \mathbf{R}\,\mathrm{holim}\, C_1 \otimes \mathbf{R}\,\mathrm{holim}\, C_2$$

of cosimplicial coalgebras. Here the tensor product is taken level-wise; the result above follows from the Eilenberg–Zilber Theorem.

To see the claim, let $C_1 : I \to c\mathscr{C}\mathscr{A}_+$ and $C_2 : I \to c\mathscr{C}\mathscr{A}_+$ be any two I-diagrams. Choose morphisms of I-diagrams

$$C_1 \to D_1, \qquad C_2 \to D_2,$$

where $D_1(i)$ and $D_2(i)$ are fibrant for all i, and $C_1(i) \to D_1(i)$ and $C_2(i) \to D_2(i)$ are weak equivalences. Then $C_1 \otimes C_2 \to D_1 \otimes D_2$ has the same properties: $D_1(i) \otimes D_2(i)$ is fibrant, since products of fibrant objects are fibrant and $C_1(i) \otimes C_2(i) \to D_1(i) \otimes D_2(i)$ is a weak equivalence. Hence, by Lemma 2.9 and the fact that holim commutes with finite products, we have weak equivalences

$$\begin{aligned}\mathbf{R}\,\mathrm{holim}(C_1 \otimes C_2) &\simeq \mathrm{holim}(D_1 \otimes D_2) \simeq \mathrm{holim}\, D_1 \otimes \mathrm{holim}\, D_2 \\ &\simeq \mathbf{R}\,\mathrm{holim}\, C_1 \otimes \mathbf{R}\,\mathrm{holim}\, C_2. \qquad \square\end{aligned}$$

3. Recollections on Dieudonnné modules

Let $\mathscr{H}\mathscr{A}_+$ be the category of graded, bicommutative Hopf algebras – hereinafter known as abelian Hopf algebras because an object in $\mathscr{H}\mathscr{A}_+$ is an abelian group object in $\mathscr{C}\mathscr{A}_+$. The point of the next two sections is to address the problem of computing $R^*_{\mathscr{C}\mathscr{A}} \lim_I H$ where $H : I \to \mathscr{H}\mathscr{A}_+$ is an I-diagram of abelian Hopf algebras. There are several advantages to this situation. One is the existence of the classification scheme known as Dieudonnné modules. Another is the natural presentation in the proof of Proposition 4.1 below. See (4.2) and Remark 4.2.

We begin by recalling Schoeller's work on Dieudonnné modules [12]. The point is that $\mathscr{H}\mathscr{A}_+$ is an abelian category with a set of projective generators. As such it is equivalent to a category of modules over a ring. Dieudonnné theory says which modules over which ring. Let us assume the ground field $k = \mathbb{F}_p$.

Definition 3.1. Suppose $p > 2$. Let $\mathscr{D}$ be the category of positively graded $\mathbb{Z}_p$ modules M equipped with homomorphisms

$$V : M_{2pk} \to M_{2k}, \quad F : M_{2k} \to M_{2pk}$$

so that

(1) $pM_{2k+1} = 0$ for all k and if $(k, p) = 1$, $p^{s+1} M_{2p^s k} = 0$;

(2) $VF(x) = px$ and $FV(y) = py$.

If $p = 2$, $\mathscr{D}$ is the category of graded $\mathbb{Z}_2$ modules with homomorphisms $V : M_{2k} \to M_k$ and $F : M_k \to M_{2k}$ so that $VF = FV = 2$ and

$$2^{s+1} M_{2^s(2t+1)} = 0.$$

Theorem 3.2 (Schoeller [12]). *There is an equivalence of categories*

$$D_*: \mathscr{H}\mathscr{A}^+ \to \mathscr{D}.$$

The category $\mathscr{D}$ is the category of *Dieudonnné modules*; if $H \in \mathscr{H}\mathscr{A}_+$, then D_*H is the associated Dieudonnné module. The homomorphisms V and F are to be taken as reflecting the Verschiebung and Frobenius (p^{th} power map), respectively.

In the sequel, will often state results for $p > 2$ and leave the case $p = 2$ implicit.

Example 3.3. The category $\mathscr{D}$ has an obvious set of projectives: indeed, for all $n > 0$, there is a module $F(n) \in \mathscr{D}$ so that there is a natural isomorphism

$$\mathrm{Hom}_{\mathscr{D}}(F(n), M) \cong M_n,$$

where M_n denotes the elements of degree n. If n is odd $F(n) = \mathbb{Z}/p\mathbb{Z}$ concentrated in degree n and if $n = 2p^s k$ with $(p, k) = 1$

$$F(n)_t \cong \begin{cases} \mathbb{Z}/p^{s+1}\mathbb{Z}, & t = 2p^i k,\ i \geq s, \\ \mathbb{Z}/p^{i+1}\mathbb{Z}, & t = 2p^i k,\ i \leq s, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$F^i: F(n)_n \to F(n)_{2p^{s+i}k}, \qquad V^i: F(n)_n \to F(n)_{2p^{s-i}k}$$

are surjective for all i.

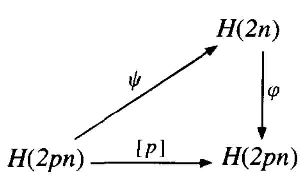
Let $H(n)$ be the unique (up to isomorphism) Hopf algebra so that $D_*H(n) \cong F(n)$. Then if n is odd $H(n) \cong \Lambda(x)$ where Λ denotes the exterior algebra and the degree of x is n. If $n = 2p^s k$ with $(p, k) = 1$,

$$H(n) \cong \mathbb{F}_p[x_0, x_1, \ldots, x_s],$$

where the degree of x_i is $2p^i k$ and $H(n)$ has the "Witt vector diagonal." The Verschiebung is described by $\xi x_i = x_{i-1}$. See [12]. Of course, the $H(n)$ are the projective generators of $\mathscr{H}\mathscr{A}_+$ and Schoeller identifies them as such before proving her theorem. Then $D_*: \mathscr{H}\mathscr{A}_+ \to \mathscr{D}$ is given, in degree n, by

$$D_n H = \mathrm{Hom}_{\mathscr{H}\mathscr{A}_+}(H(n), H).$$

The operators V and F are described by $V = \varphi^*$ and $F = \psi^*$, where $\varphi: H(2n) \to H(2pn)$ is the inclusion and $\psi: H(2pn) \to H(2n)$ is the unique map making the diagram commute



where $[p]$ is p times the identity in $\mathscr{H}\mathscr{A}_+$. In particular $\psi x_i = x_{i-1}^p$.

Example 3.4. Again assume $p > 2$. Let $n = 2k$ (with no assumption on (p, k)) and $G(n) \in \mathscr{D}$ the module with

$$G(n)_t = \begin{cases} \mathbb{Z}/p^{i+1}\mathbb{Z}, & t = 2p^i k = p^i n, \\ 0, & \text{otherwise}, \end{cases}$$

and V is surjective. Then if $n = 2p^s k$ with $(p, k) = 1$, there is a short exact sequence in $\mathscr{D}$

$$0 \to F(n) \to G(2k) \xrightarrow{V^{s+1}} G(2p^{s+1}k) \to 0.$$

If $A(n)$ is the unique (up to isomorphism) Hopf algebra with $D_* A(n) \cong G(n)$ then

$$A(n) \cong \mathbb{F}_p[x_0, x_1, x_2, \dots]$$

with $\deg(x_i) = p^i n$ and the "infinite Witt vector" diagonal. Thus there is a short exact sequence of Hopf algebras

$$\mathbb{F}_p \to H(n) \to A(2k) \to A(2p^{s+1}k) \to \mathbb{F}_p$$

if $n = 2p^s k$ with $(k, p) = 1$.

The deeper ideas about $\mathscr{H}\mathscr{A}_+$ rely on the following result. Let n_+ **Ab** be the category of positively graded abelian groups.

Proposition 3.5. *The forgetful functor $\mathscr{D} \to n_+ \mathbf{Ab}$ has a right adjoint J.*

Proof. The argument, a variant on the proof of the special adjoint functor theorem (which, by the way, would suffice in proof) is adequately covered in [9] in the context of unstable modules. So I will give only an outline. Assume $p > 2$.

Let $N \in n_+ \mathbf{Ab}$. To construct $J(N)$ it is sufficient to assume N is concentrated in a single degree, say k. Now if $J(N)$ exists, then

$$J(N)_n \cong \mathrm{Hom}_{\mathscr{D}}(F(n), J(N)) \cong \mathrm{Hom}_{\mathbf{Ab}}(F(n)_k, N_k)$$

so one simply defines

$$J(N)_n = \mathrm{Hom}_{\mathbf{Ab}}(F(n)_k, N_k).$$

The operators V and F in $\mathscr{D}$ are induced by maps $\varphi: F(2pn) \to F(2n)$ and $\psi: F(2n) \to F(2pn)$, respectively, as representing objects. These induce V and F on $J(N)$. Next notice that if $M \in \mathscr{D}$ one has maps

$$\begin{aligned} \mathrm{Hom}_{\mathscr{D}}(M, J(N)) &\to \mathrm{Hom}_{\mathbf{Ab}}(M_k, \mathrm{Hom}_{\mathbf{Ab}}(F(k)_k, N_k)) \\ &\cong \mathrm{Hom}_{\mathbf{Ab}}(M_k \otimes F(k)_k, N_k) \\ &\cong \mathrm{Hom}_{n_+\mathbf{Ab}}(M, N) \end{aligned}$$

since $M_k \to M_k \otimes F(k)_k$ is an isomorphism. By construction this map is an isomorphism if $M = F(n)$ for some n. Since both source and target send sums to products and are left exact as functions from $\mathscr{D}^{\mathrm{op}}$ to **Ab**, the result follows by analyzing projective resolutions. □

Remark 3.6. Again take $p > 2$ and let $N \in n_+ \, \mathbf{Ab}$ be concentrated in a single degree k. Then

$$J(N)_n = \mathrm{Hom}_{\mathbf{Ab}}(F(n)_k, N_k)$$

is often zero. If k is odd, $F(n)_k = 0$ unless $n = k$, thus $J(N)_n = 0$ unless $n = k$ and $J(N)_k = \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/p\mathbb{Z}, N_k)$. If $k = 2p^i k_0$, with $(k_0, p) = 1$, then $F(n)_k = 0$ unless $k = 2p^s k_0$. Hence $J(N)_n = 0$ unless $n = 2p^s k$ and

$$J(N)_{2p^s k_0} = \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/p^a\mathbb{Z}, N_k),$$

where

$$a = \begin{cases} s+1, & s \le i, \\ i+1, & s \ge i. \end{cases}$$

Corollary 3.7. *The categories $\mathscr{D}$ and $\mathscr{HA}_+$ have enough injectives.*

Proof. For the category $\mathscr{D}$ this follows from Proposition 3.5 and the fact that $n_+ \, \mathbf{Ab}$ has enough injectives. The result follows for $\mathscr{HA}_+$ because this latter category is equivalent to $\mathscr{D}$. □

Example 3.8. The functor $\mathscr{D} \to \mathbf{Ab}$ given by

$$M \mapsto \mathrm{Hom}_{\mathbf{Ab}}(M_k, \mathbb{Q}/\mathbb{Z})$$

is representable. Let $N(k) \in n_+ \, \mathbf{Ab}$ be the module isomorphic to $\mathbb{Q}/\mathbb{Z}$ in degree k. Then if $J(k) = J(N(k))$,

$$\mathrm{Hom}_{\mathscr{D}}(M, J(k)) \cong \mathrm{Hom}_{\mathbf{Ab}}(M_k, \mathbb{Q}/\mathbb{Z}).$$

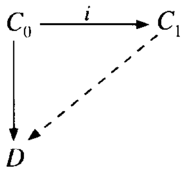
The object $J(k)$ is evidently injective since $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group. Also $J(k)$ can be described explicitly using Remark 3.6. Assume $p > 2$. If k is odd $J(k) \cong \mathbb{Z}/p\mathbb{Z}$ in degree k. If $k = 2p^i k_0$ with $(k_0, p) = 1$, then $J(k)_n = 0$ unless $n = 2p^s k_0$ and

$$J(k)_{2p^s k_0} \cong \begin{cases} \mathbb{Z}/p^{s+1}\mathbb{Z}, & s \le i, \\ \mathbb{Z}/p^{i+1}\mathbb{Z}, & s \ge i, \end{cases}$$

and V is always surjective, either by calculation or Corollary 3.15 below.

We now examine the implications for Hopf algebras.

Definition 3.9. A coalgebra $D \in \mathscr{CA}_+$ is injective if any diagram of coalgebras



with i an inclusion can be completed.

Proposition 3.10. *For a coalgebra $D \in \mathscr{CA}_+$ the following are equivalent:*
(1) $D \cong S(W)$ *for some graded vector space* W;
(2) D *is injective;*
(3) $R^s PD = 0$ *for* $s \geq 1$; *and*
(4) $R^1 PD = 0$.

Proof. That (1) implies (2) follows from adjointness. For (2) implies (3), notice that if D is injective, the inclusion $D \to \bar{S}(D)$ has a coalgebra retraction. That (3) implies (4) is clear. Finally (4) implies (1) is Proposition 4.2 of [4]. □

The connection to Hopf algebras is supplied by:

Lemma 3.11. *Let $H \in \mathscr{HA}_+$ be an injective Hopf algebra. Then H is an injective coalgebra.*

Proof. The forgetful functor $\mathscr{HA}_+ \to \mathscr{CA}_+$ has a left adjoint that preserves injections – namely, the symmetrical algebra functor endowed with the evident diagonal. Hence the forgetful functor preserves injectives. □

We now show how to recover derived functors of primitives on abelian Hopf algebras from the Dieudonnné module. If $M \in \mathscr{D}$ let $\Phi M \in \mathscr{D}$ be its "double". If $p = 2$,

$$(\Phi M)_t = \begin{cases} M_k, & t = 2k, \\ 0, & t = 2k+1, \end{cases}$$

where V and F are induced from M. If $p > 2$

$$(\Phi M)_t = \begin{cases} M_{2k}, & t = 2pk, \\ 0, & \text{otherwise}, \end{cases}$$

with again V and F induced from M. The homomorphism V induces a homomorphism in $\mathscr{D}$,

$$V : M \to \Phi M.$$

Proposition 3.12. *Let $H \in \mathscr{H}\mathscr{A}_+$. Then there is a natural exact sequence*

$$0 \to PH \to D_* H \xrightarrow{V} \Phi D_* H \to R^1 PH \to 0.$$

Proof. Assume $p > 2$. If n is odd or $n = 2k$ with $(p, k) = 1$, then $(\Phi D_* H)_n = (R^1 PH)_n = 0$, so here we are asserting $(PH)_n = D_n H$. If n is odd, one has

$$(PH)_n \cong \operatorname{Hom}_{\mathscr{H}\mathscr{A}_+}(\Lambda(x), H) \cong D_n H,$$

where $\deg(x) = n$; if $n = 2k$, $(k, p) = 1$,

$$(PH)_n \cong \operatorname{Hom}_{\mathscr{H}\mathscr{A}_+}(\mathbb{F}_p[x_0], H) \cong D_n H,$$

where the degree of x_0 is n. So we may assume $n = 2p^s k$, $s > 1$, with $(k, p) = 1$. Then there is a short exact sequence of Hopf algebras

$$\mathbb{F}_p \to H\left(\frac{n}{p}\right) \xrightarrow{\varphi} H(n) \to \mathbb{F}_p[x_s] \to \mathbb{F}_p, \tag{3.1}$$

where $H(n) \cong \mathbb{F}_p[x_0, \ldots, x_s]$ is the projective of Example 3.3 and φ is the inclusion. Since

$$\operatorname{Hom}_{\mathscr{H}\mathscr{A}_+}(H(n), H) = D_n H$$

and φ defines V we get an exact sequence

$$0 \to \operatorname{Hom}_{\mathscr{H}\mathscr{A}_+}(\mathbb{F}_p[x_s], H) \to D_n H \xrightarrow{V} D_{n/p} H \to \operatorname{Ext}^1_{\mathscr{H}\mathscr{A}_+}(\mathbb{F}_p[x_s], H) \to 0.$$

Since $\operatorname{Hom}_{\mathscr{H}\mathscr{A}_+}(\mathbb{F}_p[x_s], H) \cong (PH)_n$, the result follows from the next Lemma.

Lemma 3.13. *For all $H \in \mathscr{H}\mathscr{A}_+$ and all $m \geq 0$, there are natural isomorphisms*

$$\operatorname{Ext}^m_{\mathscr{H}\mathscr{A}_+}(\mathbb{F}_p[x_s], H) \xrightarrow{\cong} (R^m PH)_n.$$

Proof. Let $\mathbb{F}_p \to H_0 \to H_1 \to H_2 \to \mathbb{F}_p$ be a short exact sequence of Hopf algebras. Then one gets a long exact sequence in Ext* and in $R^* P(\cdot)$, the latter by Theorem 3.6 of [4]. Since $\mathscr{H}\mathscr{A}_+$ has enough injectives and $R^m PH = 0$ for $m \geq 1$ for all injectives H, by Proposition 3.10, and

$$\operatorname{Hom}_{\mathscr{H}\mathscr{A}_+}(\mathbb{F}_p[x_s], H) \cong (PH)_n$$

the result follows from standard facts about derived functors in abelian categories. See [6, p. 206], for example. □

Note that Lemma 3.13 and the two term projective resolution (3.1) immediately imply

Corollary 3.14. *For all $H \in \mathscr{H}\mathscr{A}_+$, $R^s PH = 0$ for $s \geq 2$.*

Also, Proposition 3.12 immediately implies

Corollary 3.15 *A Hopf algebra $H \in \mathscr{HA}_+$ is injective as a coalgebra if and only if $V: D_* H \to \Phi D_* H$ is surjective.*

In the future we may say V is surjective on M if $V: M \to \Phi M$ is surjective.

4. Derived functors of limits for abelian Hopf algebras

We apply the technology of the previous section to calculate $R^*_{\mathscr{CA}} \lim_I H$ for various diagrams of abelian Hopf algebras. Note that just as the forgetful functor from abelian groups to sets makes all limits of abelian groups, so too does the forgetful functor from $\mathscr{HA}_+$ to $\mathscr{CA}_+$. Thus if $H: I \to \mathscr{HA}_+$ is a diagram of Hopf algebras, the expression $\lim_I H$ is unambiguous.

Recall that $R^*_{\mathscr{CA}} \lim_I C = \pi^* \operatorname{diag} \Pi^\bullet \bar{S}^\bullet C$ is a bigraded coalgebra. The elements of $R^s_{\mathscr{CA}} \lim_I C$ will be said to have cosimplicial degree s. An element in $[R^*_{\mathscr{CA}} \lim_I C]_t$ will be said to have internal degree t.

The first result concerns products; that is, the case when I is discrete.

Proposition 4.1. *Let $\{H_\alpha\}$ be an arbitrary set of abelian Hopf algebras. Then $R^s_{\mathscr{CA}}(\Pi_\alpha) H_\alpha = 0$ for $s > 0$ and*

$$R^0_{\mathscr{CA}}\left(\prod_\alpha\right) H_\alpha = \prod_\alpha H_\alpha.$$

The proof follows after some preliminaries. Note that the statement about R^0 is automatic (See Remark 1.4(3)), so the heart of the matter is the vanishing of the higher derived functors.

To set the stage for the proof, let $D_* H_\alpha \in \mathscr{D}$ be the Dieudonné module of H_α. In $\mathscr{D}$, choose an inclusion $D_* H_\alpha \to J^0_\alpha$ where J^0_α is injective. This is possible by Corollary 3.7. Let J^1_α be the cokernel of the inclusion, so there is an exact sequence in $\mathscr{D}$

$$0 \to D_* H_\alpha \to J^0_\alpha \to J^1_\alpha \to 0. \tag{4.1}$$

Since J^0_α is injective, Lemma 3.11 and Corollary 3.15 imply $V: J^0_\alpha \to \Phi J^0_\alpha$ is surjective. Since Φ is exact, $V: J^1_\alpha \to \Phi J^1_\alpha$ is surjective. Thus (4.1) corresponds to a short exact sequence of Hopf algebras

$$\mathbb{F}_p \to H_\alpha \to K^0_\alpha \to K^1_\alpha \to \mathbb{F}_p \tag{4.2}$$

with $D_* K^0_\alpha \cong J^0_\alpha$ and $D_* K^1_\alpha \cong J^1_\alpha$ and, by Corollary 3.15, K^0_α and K^1_α are injective as coalgebras.

Proof of Proposition 4.1. The central observation is that because the sequence of Dieudonné modules

$$0 \to \prod_\alpha D_* H_\alpha \to \prod_\alpha J^0_\alpha \to \prod_\alpha J^1_\alpha \to 0$$

is exact, the sequence of Hopf algebras

$$\mathbb{F}_p \to \prod_\alpha H_\alpha \to \prod_\alpha K_\alpha^0 \to \prod_\alpha K_\alpha^1 \to \mathbb{F}_p$$

is exact. (Compare Remark 4.2 below.) We will argue that for $\varepsilon = 0, 1$

$$R^*_{\mathscr{C}\mathscr{A}}\left(\prod_\alpha K_\alpha^\varepsilon\right) \cong \prod_\alpha K_\alpha^\varepsilon \tag{4.3}$$

concentrated in cosimplicial degree 0 and then, using an argument of Bousfield's, that the result follows.

To begin, note that if we have a set of coalgebras of the form $S(V_\alpha)$ then $R^*_{\mathscr{C}\mathscr{A}}(\prod_\alpha S(V_\alpha)) \cong \prod_\alpha S(V_\alpha)$ concentrated in cosimplicial degree zero.To see this, note that the canonical resolution

$$S(V_\alpha) \to \bar{S}^\bullet S(V_\alpha)$$

has a cosimplicial concentration. Hence

$$\prod S(V_\alpha) \to \prod_\alpha \bar{S}^\bullet S(V_\alpha)$$

has a cosimplicial contradiction. Now use Example 1.6.

From this observation and Proposition 3.10, Eq. (4.3) follows.

To complete the argument, we adapt an argument of Bousfield's (pp. 477–479 of [4]). The reader who does not like the language of closed model categories can use Bousfield's explicit constructions.

Consider, for each α, the canonical resolution

$$K_\alpha^1 \to \bar{S}^\bullet K_\alpha^1.$$

Regarding K_α^1 as a constant simplicial coalgebra, this is a weak equivalence from K_α^1 to a fibrant cosimplicial coalgebra. In $c\mathscr{C}\mathscr{A}_+$, factor the composite $K_\alpha^0 \to K_\alpha^1 \to \bar{S}^\bullet K_\alpha^1$ as

$$K_\alpha^0 \xrightarrow{j} W_\alpha^\bullet \xrightarrow{q_\alpha} \bar{S}^\bullet K_\alpha^1,$$

where j is a weak equivalence and q_α is a fibration. This can be done explicitly using the cobar construction of [4]. Since compositions of fibrations are fibrations, $W_\alpha^\bullet$ is fibrant, so $j: K_\alpha^0 \to W_\alpha^\bullet$ is a resolution of K_α^0. Now let

$$\bar{W}_\alpha^\bullet = \mathbb{F}_p \square_{\bar{S}^\bullet K_\alpha^1} W_\alpha^\bullet$$

and

$$H_\alpha \cong \mathbb{F}_p \square_{K_\alpha^1} K_\alpha^0 \to \bar{W}_\alpha^\bullet. \tag{4.4}$$

the induced map. Since pull-backs of fibrations are fibrations, $\bar{W}_\alpha^\bullet$ is fibrant. Also, the pull-back diagram of cosimplicial coalgebras

$$\begin{array}{ccc} \bar{W}^\bullet_\alpha & \longrightarrow & W^\bullet_\alpha \\ \downarrow & & \downarrow{\scriptstyle q_\alpha} \\ \mathbb{F}_p & \longrightarrow & \bar{S}^\bullet K^1_\alpha \end{array}$$

yields a spectral sequence, because q_α is a fibration:

$$\mathrm{Cotor}^s_{\pi^*\bar{S}^\bullet K^1_\alpha}(\mathbb{F}_p, \pi^* W^\bullet_\alpha)_t \Rightarrow \pi^{s+t}\bar{W}^\bullet_\alpha.$$

(To see this, use [11, §II.6] where the simplicial case is considered, or adapt [4]). The degree t comes from the cosimplicial degree. Since $\pi^*\bar{S}^\bullet K^1_\alpha \cong K^1_\alpha$, $\pi^* W^\bullet_\alpha \cong K^0_\alpha$ in degree 0 and $K^0_\alpha \to K^1_\alpha$ is a surjective map of Hopf algebras, $\mathrm{Cotor}^s = 0$ for $s > 0$ and $\mathrm{Cotor}^0 \cong H_\alpha$, so (4.4) is a resolution of H_α.

Now there are isomorphisms

$$R^*_{\mathscr{C}\mathscr{A}}\left(\prod_\alpha\right) K^0_\alpha \cong \pi^*\left(\prod_\alpha W^\bullet_\alpha\right) \cong \prod_\alpha K^0_\alpha \tag{4.5a}$$

$$R^*_{\mathscr{C}\mathscr{A}}\left(\prod_\alpha\right) H_\alpha \cong \pi^*\left(\prod_\alpha \bar{W}^\bullet_\alpha\right). \tag{4.5b}$$

To see this, for example, in the second case, note there is a weak equivalence $\bar{S}^\bullet H_\alpha \to \bar{W}^\bullet_\alpha$ *in* $c\mathscr{C}\mathscr{A}_+$. Since every object of $c\mathscr{C}\mathscr{A}_+$ is cofibrant, this is a cosimplicial homotopy equivalence. Hence $\prod_\alpha \bar{S}^\bullet H_\alpha \to \prod \bar{W}^\bullet_\alpha$ is a homotopy equivalence. The second isomorphism of (4.5a) is (4.3).

There is a pull-back diagram in $c\mathscr{C}\mathscr{A}_+$

$$\begin{array}{ccc} \prod_\alpha \bar{W}^\bullet_\alpha & \longrightarrow & \prod_\alpha W^\bullet_\alpha \\ \downarrow & & \downarrow{\scriptstyle \Pi q_\alpha} \\ \mathbb{F}_p & \longrightarrow & \prod_\alpha \bar{S}^\bullet K^1_\alpha\,. \end{array}$$

Since products of fibrations are fibrations, we get, as above, a spectral sequence

$$\mathrm{Cotor}^s_{\prod_\alpha K^1_\alpha}\left(\mathbb{F}_p, \prod_\alpha K^0_\alpha\right)_t \Rightarrow \pi^{s+1}\left(\prod_\alpha \bar{W}^\bullet_\alpha\right).$$

Here we use (4.3). Since

$$\prod_\alpha K^0_\alpha \to \prod_\alpha K^1_\alpha$$

is a surjective map of Hopf algebras (this is the observation at the beginning of the proof) $\mathrm{Cotor}^s = 0$ for $s > 0$ and, in degree 0,

$$\mathbb{F}_p \,\square_{\prod_\alpha K^1_\alpha} \prod_\alpha K^0_\alpha \cong \prod_\alpha (\mathbb{F}_p \,\square_{K^1_\alpha} K^0_\alpha) \cong \prod_\alpha H_\alpha$$

as required. □

Remark 4.2. This argument, although long, is basically formal except for one point. To elucidate this, consider the situation: Bousfield defines a coalgebra to be nice if there is a sequence of coalgebras

$$\mathbb{F}_p \to C \xrightarrow{j} C^0 \xrightarrow{q} C^1 \to \mathbb{F}_p$$

so that qj is trivial, q is surjective, C^0 is an injective C^1 comodule, there are vector spaces W^0 and W^1 and isomorphisms $C^0 \cong SW^0$, $C^1 \cong SW^1$, and $C \cong \mathbb{F}_p \square_{C^1} C^0$. Given a set of nice coalgebras C_α one has such sequences

$$\mathbb{F}_p \to C_\alpha \to C_\alpha^0 \xrightarrow{q_\alpha} C_\alpha^1 \to \mathbb{F}_p$$

and can form

$$\mathbb{F}_p \to \prod C_\alpha \to \prod C_\alpha^0 \xrightarrow{\Pi q_\alpha} \Pi C_\alpha^1$$

but it is not clear that Πq_α is surjective, because of the way the product is defined. (See (1.1a) and (1.1b). So the final step of the argument breaks down. What we have done is arrange a situation where the product map is surjective.

Proposition 4.1 has the following immediate consequence.

Corollary 4.3. *Let $H: I \to \mathscr{H}\mathscr{A}_+$ be a diagram of abelian Hopf algebras. Then the augmentation*

$$\Pi^\bullet H \to diag(\Pi^\bullet \bar{S}^\bullet H)$$

induces an isomorphism

$$\pi^*(\Pi^\bullet H) \xrightarrow{\cong} R^*_{\mathscr{C}\mathscr{A}} \lim_I H.$$

Proof. The augment is exactly that of Example 1.5, once one applies Proposition 4.1 to prove

$$R^*_{\mathscr{C}\mathscr{A}}(\Pi^p)H \cong \pi^* \Pi^p \bar{S}^\bullet H \cong \Pi^p H$$

concentrated in cosimplicial degree zero. □

We use this to make a sequence of calculations. We will need the following lemma, whose proof is postponed to the next section because the argument uses more model category technology.

Lemma 4.4. *Let $H^\bullet \to K^\bullet$ be a morphism of cosimplicial abelian Hopf algebras. If*

$$\pi^* D_* H^\bullet \to \pi^* D_* K^\bullet$$

is an isomorphism, then

$$\pi^* H^\bullet \to \pi^* K^\bullet$$

is an isomorphism.

Corollary 4.5. *Let $H: I \to \mathscr{H}\mathscr{A}_+$ be a diagram of abelian Hopf algebras. If $\lim_I^s D_* H = 0$ for $s > 0$, then $R^s_{\mathscr{C}\mathscr{A}} \lim_I H = 0$ for $s > 0$ and*

$$R^0_{\mathscr{C}\mathscr{A}} \lim_I H \cong \lim_I H.$$

Note. In this statement, $\lim_I^s D_* H$ are the usual derived functors of graded abelian groups.

Proof. Again the statement about R^0 is formal. For the higher derived functors, consider the cosimplicial Dieudonné module

$$D_*(\Pi^\bullet H) \cong \Pi^\bullet D_* H.$$

Then $\pi^s \Pi^\bullet D_* H \cong \lim_I^s D_* H = 0$ if $s > 0$. Hence the augmentation

$$\lim_I D_* H \to \Pi^\bullet D_* H$$

induces a π^* isomorphism from the constant cosimplicial Dieudonné module of $\lim_I D_* H$ to $\Pi^\bullet D_* H$. Thus, by Lemma 4.4, the induced map

$$\lim_I H \to \Pi^\bullet H$$

from the constant cosimplicial object is a π^* isomorphism and

$$\lim_I H \cong \pi^*(\Pi^\bullet H) \cong R^*_{\mathscr{C}\mathscr{A}} \lim_I C$$

(using Corollary 4.3) concentrated in degree zero. □

Example 4.6. Suppose we have a tower over the natural numbers in $\mathscr{H}\mathscr{A}_+$

$$H_1 \leftarrow H_2 \leftarrow H_3 \leftarrow \cdots .$$

Then under any of the following conditions

$$R^*_{\mathscr{C}\mathscr{A}} \lim H_i \cong \lim_i H_i$$

concentrated in cosimplicial degree zero,

(1) each of the Hopf algebras is of finite type;

(2) the tower $\{H_i\}$ is pro-isomorphic, in the category of Hopf algebras, to a constant tower;

(3) the tower $\{H_i\}$ is Mittag–Leffler.

Proof. In each case $\lim^s D_* H_i = 0$ for $s > 0$. For $s > 1$, this is always true. So one need only argue $\lim^1 D_* H_i = 0$. Case by case we have:

(1) If $H \in \mathscr{H}\mathscr{A}_+$ is of finite type then $D_* H$ is of finite type. This is because $D_n H \cong \mathrm{Hom}_{\mathscr{H}\mathscr{A}_+}(H(n), H) \subseteq \mathrm{Hom}_{\mathrm{algebras}}(H(n), H)$ and $H(n)$ is a free algebra on

finitely many generators. The result now follows because $\lim^1$ vanishes on graded abelian groups of finite type.

(2) If $\{H_i\}$ is pro-isomorphic to a constant tower, so is $\{D_* H_i\}$. Hence $\lim^1 D_* H_i = 0$.

(3) Mittag–Leffler is the following condition: for fixed i, let

$$H_i^{(k)} = \operatorname{Im}\{H_{i+k} \to H_i\}.$$

Then $H_i^{(k+1)} \subseteq H_i^{(k)}$ and one demands that this descending chain stabilize: for each i, there is an N so that $H_i^{(N)} = \bigcap H_i^{(k)}$. The claim is that

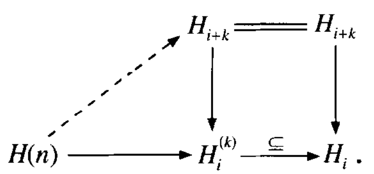
$$D_*(H_i^{(k)}) \cong \operatorname{Im}\{D_* H_{i+k} \to D_* H_i\} = (D_* H_i)^{(k)}.$$

If so, the tower of abelian groups $\{D_* H_i\}$ is Mittag–Leffler and has vanishing $\lim^1$. (See [5, §IX.3].)

To see the claim, note that since $D_n H = \operatorname{Hom}_{\mathcal{H}\mathcal{A}_*}(H(n), H)$,

$$(D_* H_i)^{(k)} \subseteq D_*(H_i^{(k)}).$$

To get the other inclusion, given a map $H(n) \to H_i^{(k)}$ one gets a diagram



The dotted arrow exists since $H_{i+k} \to H_i^{(k)}$ is surjective and $H(n)$ is projective. □

We next turn to the case where $\lim_I^s D_* H = 0$ for $s \geq 2$; that is

$$\pi^s \Pi^\bullet D_* H = 0$$

for $s \geq 2$. We indicate how to calculate $R^*_{\mathcal{C}\mathcal{A}} \lim_I H$.

First, suppose $D^\bullet$ is a cosimplicial Dieudonné module, and $H^\bullet$ a cosimplicial abelian Hopf algebra so that $D_* H^\bullet \cong D^\bullet$: Suppose

$$\pi^s D^\bullet = 0$$

for $s \geq 2$. We now state the calculation of $\pi^* H^\bullet$.

Note that $\pi^0 D^\bullet$ and $\pi^1 D^\bullet$ are Dieudonné modules and that $D_* \pi^0 H^\bullet \cong \pi^0 D^\bullet$. Define graded vector spaces $P\pi^1$ and $R^1 P\pi^1$ by the exact sequence

$$0 \to P\pi^1 \to \pi^1 D^\bullet \xrightarrow{V} \Phi \pi^1 D^\bullet \to R^1 P\pi^1 \to 0.$$

If k is a field and V a graded vector space over k which is concentrated in even degrees if the characteristic of k is not 2, let $k[V]$ be the "polynomial algebra" on V; that is, $k[V]$ is the symmetric algebra on V, which becomes isomorphic to a polynomial algebra after one chooses a basis.

Lemma 4.7. *Suppose $p > 2$ and $\pi^1 D^\bullet$ is trivial in odd degrees. Then there is an isomorphism of bigraded Hopf algebras*

$$\pi^* H^\bullet \cong \pi^0 H^\bullet \otimes \Lambda(P\pi^1) \otimes \mathbb{F}_p[R^1 P\pi^1],$$

where $P\pi^1$ and $R^1 P\pi^1$ are in cosimplicial degree 1 and 2, respectively. Furthermore, the elements of $P\pi^1$ and $R^1 P\pi^1$ are primitive.

This will be proved in the next section. If $\pi^1 D^\bullet$ is not trivial in odd degrees one can still write down the answer, but it is more complicated. The techniques of the next section suffice for the computation.

If $p = 2$, the conclusion of Lemma 4.7 holds without restrictions on $\pi^1 D^\bullet$; however, there is the possibility of an algebra extension.

Lemma 4.8. *If $p = 2$, there is a natural homomorphism*

$$\beta : P\pi^1 \to R^1 P\pi^1$$

doubling internal degree and an isomorphism of Hopf algebras

$$\pi^* H^\bullet \cong \pi^0 H^\bullet \otimes \mathbb{F}_2[P\pi^1] \otimes \mathbb{F}_2[R^1 P\pi^1]/(\beta x + x^2),$$

where $P\pi^1$ and $R^1 P\pi^1$ are in cosimplicial degree 1 and 2, respectively. Furthermore, $P\pi^1$ and $R^1 P\pi^1$ are primitive.

This also will be proved in the next section.

Example 4.9. Consider a tower of abelian Hopf algebras over the natural numbers

$$H_1 \leftarrow H_2 \leftarrow \cdots,$$

where each H_i is injective as a coalgebra and concentrated in even degrees if $p > 2$. Then we must calculate with $\lim^1 D_* H_i$. However, there is an exact sequence

$$0 \to \lim D_* H_i \to \prod D_* H_i \to \prod D_* H_i \to \lim^1 D_* H_i \to 0.$$

Since V is surjective on $D_* H_i$ for all i (by Corollary 3.15) it is surjective on $\lim^1 D_* H_i$. Hence $R^1 P\pi^1 = 0$ and

$$R^*_{\mathscr{C}\mathscr{A}} \lim H_i = \lim H_i \otimes \Lambda(P \lim^1 D_* H_i).$$

Furthermore, treating $\Pi D_* H_i \to \Pi D_* H_i$ is a two term cochain complex, one gets a short exact sequence

$$0 \to R^1 P(\lim H_i) \to \lim^1 PH_i \to P \lim^1 D_* H_i \to 0.$$

Hence if $R^1 P(\lim H_i) = 0$, or equivalently, if $\lim H_i$ is an injective coalgebra, $\lim^1 PH_i \cong P \lim^1 D_* H_i$.

To be specific, let $H_i = \Gamma_p(x_i, x_{i+1}, \dots)$ be the divided algebra on infinitely many generators of degree $2t$ for some t, and let $H_{i+1} \to H_i$ be the inclusion. Then $\lim H_i = \mathbb{F}_p$ and

$$\lim{}^1 PH_i = \prod_{i=1}^{\infty} \mathbb{F}_p \Big/ \bigoplus_{i=1}^{\infty} \mathbb{F}_p.$$

Hence

$$R^*_{\mathscr{CA}} \lim H_i \cong \Lambda\left(\prod_{i=1}^{\infty} \mathbb{F}_p \Big/ \bigoplus_{i=1}^{\infty} \mathbb{F}_p \right)$$

with the generators in cosimplicial degree 1 and internal degree $2t$. Note that $R^s_{\mathscr{CA}} \lim H_i \neq 0$ for all $s \geq 1$.

Example 4.10. Let $p > 2$ and $H_i = \mathbb{F}_p[x_i, x_{i+1}, \dots]/(x_i^p, x_{i+1}^p, \dots)$ be the truncated polynomial algebra on primitive generators of degree $2t$ and $H_{i+1} \to H_i$ the inclusion. Then if $W = \prod_{i=1}^{\infty} \mathbb{F}_p / \bigoplus_{i=1}^{\infty} \mathbb{F}_p$ in degree $2t$, then

$$R^*_{\mathscr{CA}} \lim H_i = \Lambda(W) \otimes \mathbb{F}_p[\Phi W]$$

with W and ΦW in cosimplicial degrees 1 and 2, respectively. The same example at $p = 2$ yields, with W in cosimplicial degree 1,

$$R^*_{\mathscr{CA}} \lim H_i = \mathbb{F}_2[W].$$

Note in this case $PR^*_{\mathscr{CA}} \lim H_i$ is non-zero in infinitely many cosimplicial degrees.

Example 4.11. Let G be a free group on finitely many generators and $H \in \mathscr{HA}_+$ a G-Hopf algebra; that is G acts on H through morphisms in $\mathscr{HA}_+$. Then $D_* H$ is a G-module and

$$\lim D_* H = H^0(G, D_* H) = (D_* H)^G \quad \text{and} \quad \lim{}^s D_* H = H^s(G, D_* H)$$

which is zero for $s > 1$. Finally, $H^*(G, D_* H)$ can be calculated by an exact sequence

$$0 \to H^0(G, D_* H) \to D_* H \xrightarrow{\partial} \prod_{i=1}^{n} D_* H \to H^1(G, D_* H) \to 0, \tag{4.6}$$

where, if $\tau_i \in G$, $1 \leq i \leq n$, are the generators

$$\partial(x) = (x - \tau_1 x, \dots, x - \tau_n x).$$

Let us confuse G with category with one object and G as morphisms. Note that in $\mathscr{CA}_+$ the inclusion $\lim_G H \subset H^G$ is usually strict.

Assume $p > 2$ and that H is concentrated in even degrees. Then

$$R^*_{\mathscr{CA}} \lim_G H \cong \lim_G H \otimes \Lambda(PH^1(G, D_* H)) \otimes \mathbb{F}_p[R^1 PH^1(G, D_* H)].$$

If H is injective as a coalgebra (e.g., H_*BU), then V is surjective on D_*H, so on $H^1(G, D_*H)$ by (4.6) and

$$R^*_{\mathscr{C},\mathscr{A}} \lim_G H \cong \lim_G H \otimes \Lambda(PH^1(G, D_*H)).$$

The latter formula holds at $p = 2$ if H is injective as a coalgebra, but not necessarily in even degrees.

We finish this example by showing that $R^*_{\mathscr{C},\mathscr{A}} \lim_G H$ can be identified as a Cotor. Fix generators $\tau_1, \ldots, \tau_n$ of G and define two maps

$$H \to \underbrace{H\otimes \cdots \otimes H}_{n+1} = H^{n+1}$$

as follows. The first is $\Delta_{n+1}: H \to H^{n+1}$, the $(n+1)$st iterated diagonal. The second is the composite

$$H \xrightarrow{\Delta_{n+1}} H^{n+1} \xrightarrow{1\otimes\tau_1\otimes\cdots\otimes\tau_n} H^{n+1}.$$

Both of these maps make H into an H^{n+1} comodule. Denote the first comodule structure by H, the second by H_τ.

Proposition 4.12. *Let H be a connected G-abelian Hopf algebra. Then there is a natural isomorphism of bigraded Hopf algebras*

$$R^*_{\mathscr{C},\mathscr{A}} \lim_G H \cong \mathrm{Cotor}_{H^{n+1}}(H, H_\tau).$$

Proof. Let $B^\bullet(H) = B^\bullet(H, H^{n+1}, H_\tau)$ be the cobar construction, we will show there is a natural map

$$\Pi^\bullet D_*H = D_*\Pi^\bullet H \to D_*B^\bullet(H) \tag{4.7a}$$

giving an isomorphism on cohomotopy. The result will then follow from Corollary 4.3 and Lemma 4.4.

If M is any G-module, then one defines

$$H^s(G, M) = \lim_G{}^s M = \pi^s \Pi^\bullet M.$$

However, one can calculate $H^*(G, M)$ as the cohomology of the two stage cochain complex

$$T^*(M) = \{M \times M \xrightarrow{\partial} \underbrace{M \times \cdots \times M}_{n+1}\},$$

when $\partial(x, y) = (x - y, x - \tau_1 y, \ldots, x - \tau_n y)$. Indeed, $H^0T^*(M)^G \cong M^G$ and $H^sT^*(M) = 0$ for $s > 0$ and M an injective G-module, so there must be a natural map of cochain complexes

$$N\Pi^{\bullet}M \to T^{*}(M) \tag{4.7b}$$

inducing an isomorphism on cohomology. (This is a standard fact about derived functors; see [6, §3.1]). Now, one easily checks that $ND_{*}B^{\bullet}(H) \cong T^{*}D_{*}H$. Hence, the normalized cochain equivalence of (4.7b) gives an (unnormalized) cohomotopy equivalence in (4.7a) and the result follows. □

5. From cosimplicial Dieduonné modules to cosimplicial Hopf algebras

In this section we prove those technical results needed in Section 4 on passing from $\pi^{*}D_{*}H^{\bullet}$ to $\pi^{*}H^{\bullet}$ where $H^{\bullet}$ is a cosimplicial abelian Hopf algebra. We end with some generalities.

The first result we used was Lemma 4.4, which we record now as:

Proposition 5.1. *Let $H^{\bullet} \to K^{\bullet}$ be a morphism of cosimplicial abelian Hopf algebras. If*

$$\pi^{*}D_{*}H^{\bullet} \to \pi^{*}D_{*}K^{\bullet}$$

is an isomorphism, then $\pi^{*}H^{\bullet} \to \pi^{*}K^{\bullet}$ is an isomorphism.

This is proved by combining Lemmas 5.3, 5.4 and 5.5 below with some model category techniques. The argument is given after Lemma 5.5.

Because $\mathscr{D}$ has enough injectives there is a simplicial model category structure on the category $c\mathscr{D}$. Indeed, the simplicial structure is the obvious analog of that given for $c\mathscr{C}\mathscr{A}_{+}$ in (2.1).

Definition 5.2. A morphism $f: A^{\bullet} \to B^{\bullet}$ in $c\mathscr{D}$ is

(1) a weak equivalence if it is a π^{*} isomorphism;

(2) a cofibration if the normalized chain complex $Nf: NA^{\bullet} \to NB^{\bullet}$ is an injective in positive degrees; and

(3) a fibration if it is a surjection and $K^{\bullet} = \ker\{f: A^{\bullet} \to B^{\bullet}\}$ is levelwise an injective object.

The proof is entirely standard; see [2, Proposition 6.5] or [11, §II.4] and use the fact that the opposite category of $c\mathscr{D}$ is the category of simplicial objects in $\mathscr{D}^{\mathrm{op}}$ and $\mathscr{D}^{\mathrm{op}}$ is an abelian category with enough projectives.

Lemma 5.3. *Suppose $H^{\bullet} \to K^{\bullet}$ is a morphism of cosimplicial abelian Hopf algebras. If $D_{*}H^{\bullet}$ and $D_{*}K^{\bullet}$ are fibrant in $c\mathscr{D}$ and $\pi^{*}D_{*}H^{\bullet} \to \pi^{*}D_{*}K^{\bullet}$ is an isomorphism, then $\pi^{*}H^{\bullet} \to \pi^{*}K^{\bullet}$ is an isomorphism.*

Proof. Every object of $c\mathcal{D}$ is cofibrant so a weak equivalence of fibrant objects is a homotopy equivalence. Hence $H^\bullet \to K^\bullet$ is a cosimplicial homotopy equivalence. $\square$

Lemma 5.4. *Let $H^{-1} \to H^\bullet$ be an augmented cosimplicial abelian Hopf algebra and suppose $\pi^* D_* H^\bullet \cong D_* H^{-1}$ concentrated in cosimplicial degree 0. Then $\pi^* H^\bullet \cong H^{-1}$.*

Proof. We will define sub-cosimplicial Hopf algebras

$$H^{-1} = H(-1) \subseteq H(0) \subseteq H(1) \subseteq \cdots \subseteq H^\bullet$$

so that $\pi^* H(n) \cong \pi^* H(n+1)$ for all n and $H(n)^s = H(n+1)^s$ for all $s \leq n$. The result follows.

To define $H(n)$ consider the normalized augmented cochain complex of Dieudonné modules

$$0 \to D_* H^{-1} \to ND_* H^0 \xrightarrow{\partial} ND_* H^1 \xrightarrow{\partial} ND_* H^2 \to \cdots .$$

Let $K^n = \ker(\partial: ND_* H^n \to ND_* H^{n+1}) \cong \operatorname{Im}(\partial: ND_* H^{n-1} \to ND_* H^n)$ and let $H(n) \in c\mathcal{HA}_+$ be the augmented object with augmented normalized Dieudonné module

$$0 \to DH_*^{-1} \to ND_* H^0 \to ND_* H^1 \to \cdots \to ND_* H^n \to K^{n+1} \to 0 \to \cdots .$$

Then there is a pull-back diagram in $c\mathcal{HA}_+$, $n \geq 0$,

$$\begin{array}{ccc} H(n-1) & \longrightarrow & H(n) \\ \downarrow & & \downarrow \\ \mathbb{F}_p & \longrightarrow & K(n) \end{array}$$

where $K(n) \in c\mathcal{HA}_+$ is the object with normalized Dieudonné module

$$\cdots 0 \to K^{n+1} \to K^{n+1} \to 0 \cdots \tag{5.1}$$

in degrees n and $n+1$. Thus $H^{-1} = H(-1)$, and there is a spectral sequence

$$\operatorname{Cotor}^*_{\pi^* K(n)}(\mathbb{F}_p, \pi^* H(n)) \Rightarrow \pi^* H(n-1).$$

Now the chain complex of (5.1) has a contraction, so $K(n)$ has a cosimplicial contraction and the $\pi^* K(n) \cong \mathbb{F}_p$. $\square$

Lemma 5.5 *Let $H^\bullet \in c\mathcal{HA}_+$ be a cosimplicial abelian Hopf algebra. Then there is a natural map $\varepsilon: H^\bullet \to H_1^\bullet$ in $c\mathcal{HA}_+$ so that $\pi^*\varepsilon$ and $\pi^* D_*\varepsilon$ are isomorphisms and $D_* H_1^\bullet$ is fibrant as an object in $c\mathcal{D}$.*

Proof. For $M \in \mathscr{D}$, let (with $J(n)$ as in Example 3.8)

$$J(M) = \prod_{n} \prod_{x \in \mathrm{Hom}(M_n, \mathbb{Q}/\mathbb{Z})} J(n)$$

and let $M \to J(M)$ be the evident inclusion. Then for $M \in \mathscr{D}$ there is a natural injective resolution

$$M \to J^0(M) \to J^1(M) \to \cdots$$

with $J^0(M) = J(M)$ and $J^{n+1}(M) = J(\operatorname{coker} J^{n-1}(M) \to J^n(M))$. Thus in $H \in \mathscr{HA}_+$ one gets a natural cosimplicial resolution in $c\mathscr{HA}_+$

$$H \to J(H)^\bullet,$$

where the normalized Dieudonné module of $J(H)^\bullet$ is $J^*(D_* H)$. By Lemma 5.4, $H \cong \pi^* J(H)$. If $H^\bullet \in c\mathscr{HA}_+$ let $J(H^\bullet)^\bullet$ be the obvious bicosimplicial object and set $H_1^\bullet = \operatorname{diag} J(H^\bullet)^\bullet$. A bicomplex argument shows that $\varepsilon: H^\bullet \to H_1^\bullet$ is a π^* and $\pi^* D_*$ isomorphism. Since $D_* H_1^\bullet$ is a product of injectives at each level, $D_* H_1^\bullet$ is fibrant by the Definition 5.2(3). □

Proof of Proposition 5.1. Use Lemma 5.5 to produce a diagram

$$\begin{array}{ccc} H^\bullet & \xrightarrow{f} & K^\bullet \\ \downarrow{\scriptstyle \varepsilon_H} & & \downarrow{\scriptstyle \varepsilon_K} \\ H_1^\bullet & \xrightarrow{f_1} & K_1^\bullet \end{array}$$

so that the conclusions of Lemma 5.5 hold. Now $D_* f$, $D_* \varepsilon_H$, and $D_* \varepsilon_K$ are all π^* isomorphisms, so $D_* f_1$ is a π^* isomorphism. Hence, by Lemmas 5.3 and 5.5, $\pi^* f_1$ is an isomorphism. Since $\pi^* \varepsilon_H$ and $\pi^* \varepsilon_K$ are isomorphisms, so is $\pi^* f$. □

The next project is to calculate $\pi^* H^\bullet$, where $\pi^s D_* H^\bullet = 0$ for $s > 1$. By Proposition 5.1 we may assume $N D_* H^s = 0$ for $s > 1$, simply by letting

$$H(1)^\bullet \subseteq H^\bullet$$

be the sub-cosimplicial Hopf algebra with

$$N D_* H(1)^s = \begin{cases} N D_* H^0, & s = 0, \\ \ker(N D_* H^1 \to N D_* H^2), & s = 1, \\ 0, & s \geq 2. \end{cases}$$

Then $\pi^* D_* H(1)^\bullet \cong \pi^* D_* H^\bullet$, so $\pi^* H(1)^\bullet \cong \pi^* H^\bullet$. For short, let us write D^s for $N D_* H^s$.

Then there is a pull-back diagram of cochain complexes

$$
\begin{array}{ccc}
ND_*H^\bullet & \longrightarrow & \{D^1 \to D^1 \to 0 \to \cdots\} \\
\downarrow & & \downarrow \\
\{D^0 \to 0 \cdots\} & \longrightarrow & \{D^1 \to 0 \to \cdots\}
\end{array}
$$

Hence, there is an associated pull-back diagram of cosimplicial Hopf algebras

$$
\begin{array}{ccc}
H^\bullet & \longrightarrow & A^0 \\
\downarrow & & \downarrow \\
H^0 & \longrightarrow & A^1.
\end{array}
$$

Therefore, we get a spectral sequence

$$E_2^{s,t} = \mathrm{Cotor}^s_{A^1}(\mathbb{F}_p, H^0)^t \Rightarrow \pi^{s+t} H^\bullet,$$

where A^1 is the Hopf algebra so that $D_* A^1 \cong D^1$. The grading t arises from the cosimplicial degree on these cosimplicial Hopf algebras. Since A^1 and H^0 are both in cosimplicial degree 0, the spectral sequence collapses. Furthermore, the obvious change of rings implies

$$\pi^* H^\bullet \cong \mathrm{Cotor}^*_{A^1}(\mathbb{F}_p, H^0) \cong \pi^0 H^\bullet \otimes \mathrm{Cotor}^*_{H^1}(\mathbb{F}_p, \mathbb{F}_p), \tag{5.2}$$

where H^1 is the Hopf algebra, so that $D_* H^1 \cong \pi^1 D_* H^\bullet$.

We now show how to calculate $\mathrm{Cotor}^*_{H_1}(\mathbb{F}_p, \mathbb{F}_o)$ as a functor of

$$PH_1 \cong P\pi^1 D_* H^\bullet, \quad \text{and} \quad R^1 PH_1 \cong R^1 P\pi^1 D_* H^\bullet. \tag{5.3}$$

Choose an exact sequence of Dieudonné modules

$$0 \to D_* H_1 \to J^0 \to J^1 \to 0,$$

where V is surjective on J^0 and J^1. This can be done because $\mathcal{D}$ has enough injectives and V is surjective on injectives. This corresponds to a short exact sequence of Hopf algebras

$$\mathbb{F}_p \to H_1 \to K^0 \to K^1 \to \mathbb{F}_p,$$

where K^i, $i = 0, 1$, are injective as coalgebras. Let $B^\bullet(\cdot)$ denote the cobar construction, so that

$$\pi^* B^\bullet(K) = \mathrm{Cotor}^*_K(\mathbb{F}_p, \mathbb{F}_p).$$

Then we have a pull-back diagram of cosimplicial Hopf algebras

$$\begin{array}{ccc} B^\bullet(H_1) & \longrightarrow & B^\bullet(K^0) \\ \downarrow & & \downarrow \\ \mathbb{F}_p & \longrightarrow & B^\bullet(K^1) \end{array}$$

and hence a spectral sequence

$$E_2^{s,t} = \mathrm{Cotor}^s_{\pi^* B^\bullet(K^1)}(\mathbb{F}_p, \pi^* B^\bullet(K^0))^t \Rightarrow \mathrm{Cotor}^{s+t}_{H_1}(\mathbb{F}_p, \mathbb{F}_p). \tag{5.4}$$

If $K \in \mathscr{H}\mathscr{A}_+$ is injective as a coalgebra, $K \cong S(W)$ for some graded vector space W and (where, if $p > 2$, W is concentrated is even degrees)

$$\pi^* B^\bullet(K) \cong \mathrm{Cotor}^*_K(\mathbb{F}_p, \mathbb{F}_p) \cong \Lambda(PK).$$

This isomorphism is natural in maps of injective coalgebras. Thus we can write

$$E_2 \cong \mathrm{Cotor}^*_{\Lambda(PK^1)}(\mathbb{F}_p, \Lambda(PK^0)).$$

By a change of rings since is isomorphic to

$$\Lambda(PH_1) \otimes \mathrm{Cotor}^*_{\Lambda(R^1 PH_1)}(\mathbb{F}_p, \mathbb{F}_p) \cong \Lambda(PH_1) \otimes \mathbb{F}_p[R^1 PH_1] \tag{5.5}$$

since there is an exact sequence

$$0 \to PH_1 \to PK^0 \to PK^1 \to R^1 PH_1 \to 0.$$

Since (5.3) is a spectral sequence of Hopf algebras it must collapse, and there can be no coalgebra extensions. Thus we can give

Proof of Lemma 4.7. Combine (5.2), (5.3), and (5.5) with the observations that if $p > 2$ there can be no algebra extensions. □

If $p = 2$ there can be an algebra extension in the spectral sequence of (5.4). For example, if H_1 is an exterior algebra on a single generator, then $\mathrm{Cotor}_{H_1}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[x]$. This example is generic, and is detected by an operation $\beta: PH_1 \to R^1 PH_2$.

To define β consider a short exact sequence of Dieudonné modules

$$0 \to D_* H_1 \xrightarrow{\varepsilon} J^0 \xrightarrow{q} J^1 \to 0,$$

where V is surjective on J^0. Let $x \in PH_1 = \ker\{V: D_* H_1 \to \Phi D_* H_1\}$. Then there is a $y \in J^0$ so that $Vy = x$. The element

$$q(y) \in PH^1 = \ker\{V: J^1 \to \Phi J^1\}.$$

Then $\beta(x)$ is the image of $q(y)$ in $R^1 PH_1$ in the exact sequence

$$0 \to PH_1 \to PH^0 \to PJ^1 \to R^1 PH_1 \to 0.$$

It is an easy exercise that this is independent of the choices and $\beta(x) = 0$ if and only if x is in the image of V in $D_* H_1$.

The proof of Lemma 4.8. Because the spectral sequence (5.4) collapses and there can be no coalgebra extensions, the result follows from (5.2), (5.3) and (5.5) once we determine the algebra extensions. Since $\mathrm{Cotor}_{H_1}(\mathbb{F}_p, \mathbb{F}_p)$ commutes with filtered colimits, we may assume H_1 is of finite type. Thus

$$\mathrm{Cotor}_{H_1}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{H_1^*}(\mathbb{F}_2, \mathbb{F}_2).$$

Use Borel's structure theorem to write an algebra decomposition

$$\mathrm{H}_1^* \cong \bigotimes_i \mathbb{F}_2[x_i]/(x_i^{2n_i}),$$

where $1 \le n_i \le \infty$. Since an algebra decomposition of this sort yields a tensor product on Ext, we may assume

$$H_1^* \cong \mathbb{F}_2(x)/(x^{2^n})$$

with $1 \le n \le \infty$. But, if $\deg(x) = t$, we have

$$(D_* H)_k \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & k = 2^i t, \quad 0 \le i \le n-1, \\ 0, & \text{otherwise}, \end{cases}$$

and V is surjective. But since

$$\mathrm{Cotor}_{H_1}(\mathbb{F}_2, \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2[PH_1], & n = 1 \\ \Lambda(PH_1) \otimes \mathbb{F}_2[R^1 PH_1], & n > 1, \end{cases}$$

the result follows. □

6. The homology spectral sequence

This section joins algebra to homotopy theory and writes down a spectral sequence passing from $R^*_{\mathscr{CA}} \lim_I H_* X$ to $H_* \operatorname{holim}_I X$. Convergence is discussed and examples are given.

Recall that Bousfield and Kan [5, Chap. XI] define homotopy inverse limits in the following manner. Let $X : I \to \mathbf{S}$ be an I-diagram of spaces. Then one has the associated cosimplicial space $\Pi^\bullet X$ and

$$\operatorname{holim} X = \operatorname*{holim}_I X = \mathrm{Tot}(\Pi^\bullet X). \tag{6.1}$$

If each $X(i)$ is fibrant, then $\Pi^\bullet X$ is a fibrant cosimplicial space, and the associated homotopy spectral sequence (for a pointed diagram)

$$\pi^s \pi_t \Pi^\bullet X = \lim{}^s \pi_t X \Rightarrow \pi_{t-s} \operatorname{holim} X \tag{6.2}$$

is the expected one. If some $X(i)$ is not fibrant, one probably does not want to take its homotopy inverse limit. Rather, one should take a morphism of diagrams $X \to Y$, where each $X(i) \to Y(i)$ is a weak equivalence with $Y(i)$ fibrant and simply define $\operatorname{holim} X = \operatorname{holim} Y$. This is analogous to what occurred in Section 2 or, in this case, in [5, §XI.5]. Then (6.2) holds regardless.

Associated to the cosimplicial space $\Pi^\bullet X$ is a homology spectral sequence

$$\pi^s H_t(\Pi^\bullet X) \Rightarrow H_{t-s}(\Pi^\bullet X).$$

This spectral sequence, due to Anderson [1], is discussed in [3, 4.3]; however, since $\Pi^n X$ is, in general, an infinite product, this spectral sequence does not have an accessible E_1 term. Hence we propose a modification.

For a space X, let η: $X \to \mathbb{F}_p^\bullet X$ be the Bousfield–Kan resolution X. It is the cosimplicial resolution of X derived from the triple on space sending a simplicial set X to the simplicial vector space $\mathbb{F}_p X$ generated by X. (Actually, this is Lannes's variant [8, §1.5] of the Bousfield–Kan resolution.) Then the Bousfield–Kan p-completion on X is defined by

$$X_p = \operatorname{Tot}(\mathbb{F}_p^\bullet X).$$

Now if $X : I \to \mathbf{S}$ is an I-diagram, $\mathbb{F}_p^\bullet X$ is an I-diagram of cosimplicial spaces. Thus we can form the bicosimplicial space

$$\Pi^\bullet \mathbb{F}_p^\bullet X$$

and its diagonal $\operatorname{diag}(\Pi^\bullet \mathbb{F}_p^\bullet X)$. That this is a good thing to do is indicated by the following two results.

Lemma 6.1. *The cosimplicial space* $\operatorname{diag}(\Pi^\bullet \mathbb{F}_p^\bullet X)$ *is fibrant and*

$$\operatorname{Tot}(\operatorname{diag} \Pi^\bullet \mathbb{F}_p^\bullet X) \cong \operatorname{holim} X_p,$$

the homotopy inverse limit of the p-completions.

Proof. The first claim is proved exactly as in [13, Lemma 7.1]. For the second, write $\Pi^\bullet \mathbb{F}_p^\bullet = \{\Pi^n \mathbb{F}_p^m X\}$ and call n and m the horizontal and vertical directions, respectively. Then by [13, Lemma 7.8]

$$\operatorname{Tot}(\operatorname{diag} \Pi^\bullet \mathbb{F}_p^\bullet X) \cong \operatorname{Tot}^h \operatorname{Tot}^v(\Pi^\bullet \mathbb{F}_p^\bullet X)$$

where h and v means take Tot in the horizontal or vertical directions. Since Tot commutes with products,

$$\operatorname{Tot}^v(\Pi^n \mathbb{F}_p^\bullet X) \cong \Pi^n X_p$$

and the results follows. □

Lemma 6.2 *Suppose $X : I \to \mathbf{S}$ is an I-diagram of connected spaces. There is a natural isomorphism of bigraded coalgebras*

$$\pi^* H_* \operatorname{diag} \bar{\mathbb{F}}_p^\bullet X^\bullet \cong R^*_{\mathscr{C}\mathscr{A}} \lim H_* X .$$

Proof. This is a direct application of the results of Section 2. Regard $H_* X : I \to c\mathscr{C}\mathscr{A}_+$ as a constant I-diagram of cosimplicial coalgebras. Then

$$H^* X \to H_* \mathbb{F}_p^\bullet X$$

is a morphism of I-diagrams in $c\mathscr{C}\mathscr{A}_+$ and $H_* X(i) \cong \pi^* H_* \mathbb{F}_p^\bullet X(i)$ for all i. Furthermore, for any space connected space Y, there is a functorial isomorphism, with $W(Y)$ a vector space functorial in Y,

$$H_* \mathbb{F}_p Y \cong S(W(Y))$$

because $\mathbb{F}_p Y$ is a mod-p generalized Eilenberg–MacLane space. Hence $H_* \mathbb{F}_p^\bullet X(i)$ is almost-free, hence fibrant and, by Lemma 2.9,

$$\begin{aligned} R^*_{\mathscr{C}\mathscr{A}} \lim H_* X &\cong \pi^* \operatorname{diag} \Pi^\bullet H_* \mathbb{F}_p^\bullet X \\ &\cong \pi^* H_* (\operatorname{diag} \Pi^\bullet \mathbb{F}_p^\bullet X). \qquad \square \end{aligned}$$

Thus we have

Proposition 6.3. *There is a second-quadrant homology spectral sequence*

$$[R^s_{\mathscr{C}\mathscr{A}} \lim_I H_* X]_t \Rightarrow H_{t-s} \operatorname{holim}_I X_p .$$

Proof. This is the homology spectral sequence

$$\pi^s H_t \operatorname{diag} \Pi^\bullet \mathbb{F}_p^\bullet X \Rightarrow H_{t-s} \operatorname{Tot}(\operatorname{diag} \Pi^\bullet \mathbb{F}_p^\bullet X)$$

using Lemmas 6.1 and 6.2. $\square$

6.4. Note on convergence. Because each of the spaces $\Pi^n \mathbb{F}_p^n X$ is simple and p-nilpotent, any of Bousfield's Theorems 3.2, 3.4, or 3.6 will apply to give strong convergence. For example, we will have strong convergence if $X : I \to \mathscr{C}\mathscr{A}_+$ is a pointed I-diagram and either

(1) $\pi^s \pi_t \operatorname{diag} \mathbb{F}_p^\bullet X = 0$ for $t - s \leq 0$ and for each s there are only finitely many n so that $\pi^s \pi_{s+n} \operatorname{diag} \mathbb{F}_p^\bullet X \neq 0$, or

(2) If $[R^0_{\mathscr{C}\mathscr{A}} \lim H_* X]_0 \cong \mathbb{F}_p$, $[R^s_{\mathscr{C}\mathscr{A}} \lim H_* X]_{st} = 0$ for $t - s \leq 1$ and for each s there are only finitely many n so that

$$[R^s_{\mathscr{C}\mathscr{A}} \lim H_*]_{s+n} \neq 0 .$$

There are other convergence results in [3] and [13].

We devote the rest of this section to examples.

Example 6.5. We prove here that if $\{X_\alpha\}$ is a set of connected nilpotent spaces with, for all α, H_*X_α the coalgebra underlying an abelian Hopf algebra, then

$$H_*\left(\prod_\alpha X_\alpha\right) \cong \prod H_*X_\alpha \cong H_*\left(\prod_\alpha (X_\alpha)_p\right),$$

where the product on the middle is in $\mathscr{C}\mathscr{A}_+$.

Indeed, if $\prod_\alpha \mathbb{F}_p^\bullet X_\alpha \to \operatorname{diag}(\Pi^\bullet \mathbb{F}_p^\bullet X)$ is the augmentation, we get a diagram of spectral sequences

$$\begin{array}{ccc} \pi^* H_*(\prod_\alpha \mathbb{F}_p^\bullet X_\alpha) & \Longrightarrow & H_*(\operatorname{Tot}(\prod_\alpha \mathbb{F}_p^\bullet X_\alpha)) \\ \downarrow & & \downarrow \\ R^*_{\mathscr{C}\mathscr{A}}(\prod_\alpha) H_* X_\alpha \cong \pi^* H_*(\operatorname{diag} \Pi^\bullet \mathbb{F}_p^\bullet X) & \Rightarrow & H_*(\prod_\alpha (X_\alpha)_p). \end{array} \tag{6.3}$$

As in Example 1.6, the map on E_2 terms is an isomorphism. Since Tot commutes with products, $\operatorname{Tot}(\prod_\alpha \mathbb{F}_p^\bullet X_\alpha) \cong \prod_\alpha (X_\alpha)_p$, so we have isomorphic spectral sequences. But in [3, 4.14] it is proved that this spectral sequence converges completely to $H_*(\prod_\alpha X_\alpha)$. By Proposition 4.1,

$$R^*_{\mathscr{C}\mathscr{A}}\left(\prod_\alpha\right) H_*X_\alpha \cong \prod_\alpha H_*X_\alpha$$

concentrated in cosimplicial degree zero. This yields one of the isomorphisms. On the other hand, since $(X_\alpha)_p$ is also nilpotent

$$H_*\left(\prod_\alpha (X_\alpha)_p\right) \cong \prod_\alpha H_*(X_\alpha)_p \cong \prod_\infty H_*X_\alpha.$$

Note that we have proved that the isomorphic spectral sequences of (6.3) converge strongly.

Example 6.6. Let $X: I \to \mathbf{S}$ be an I-diagram of pointed, connected simple spaces and suppose

(1) for all $i \in I$, $H_*X(i)$ is the coalgebra underlying an abelian Hopf algebra and

(2) $\lim^s \pi_t X = 0$ for $t - s \leq 0$ and for s there are only finitely many n so that $\lim^s \pi_{s+n} X \neq 0$.

Then there is a strongly convergent homology spectral sequence

$$R^*_{\mathscr{C}\mathscr{A}} \lim_I H_*X \Rightarrow H_* \operatorname{holim}_I X.$$

Note that (2) will happen if each $X(i)$ is simply connected and I diagram is a tower over the natural numbers or a space with an action by a free group.

To see this, consider the augmented bicosimplicial space

$$\Pi^\bullet X \to \Pi^\bullet \mathbb{F}_p^\bullet X.$$

Then one gets a diagram of homology spectral sequences

$$\begin{array}{ccc} \pi^* H_*(\Pi^\bullet X) & \Longrightarrow & H_* \operatorname{holim} X \\ \downarrow & & \downarrow \\ R^*_{\mathscr{CA}} \lim\limits_I H_* X \cong \pi^* H_* \operatorname{diag}(\Pi^\bullet \mathbb{F}_p^\bullet X) & \Rightarrow & H_* \operatorname{holim} X_p. \end{array} \tag{6.4}$$

A bicomplex argument and Example 6.5 show that the map on E_2 terms is an isomorphism. Strong convergence for the top spectral sequence follows from 6.4(1). Note if hypothesis 6.6(2) also holds for the I-diagram

$$X_p : I \to \mathbf{S}$$

then the lower spectral sequence of (6.4) also converges. This follows from the fact that $H_* X \cong H_* X_p$ and the diagram

$$\begin{array}{ccc} \pi^* H_*(\Pi^\bullet X_p) & \Rightarrow & H_* \operatorname{holim} X_p \\ \downarrow \cong & & \downarrow \cong \\ R^*_{\mathscr{CA}} \lim\limits_I H_* X \cong \pi^* H_* \operatorname{diag}(\Pi^\bullet \mathbb{F}_p^\bullet X_p) & \Rightarrow & H_* \operatorname{holim} X_p \end{array}$$

obtained from (6.4) is an isomorphism of spectral sequences.

Example 6.7. Let

$$X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$$

be a tower of connected spaces so that the resulting diagram $H_* X : I \to \mathscr{CA}_+$ factors through $\mathscr{HA}_+$; in short, one has a tower of abelian Hopf algebras $\{H_* X_n\}$. Suppose this tower of coalgebras is either of finite type, Mittag–Leffler, or pro-isomorphic to a constant tower. Then by Example 4.6, $R^*_{\mathscr{CA}} \lim_i H_* X_i \cong \lim_i H_* X_i$ concentrated in simplicial degree 0. As throughout this paper, the limit at the right is in the category of coalgebras. Thus if each of the spaces is simple and $\lim^1 \pi_1 X_i = 0$ we have

$$H_* \operatorname{holim}_i X_i \cong \lim_i H_* X_i, \tag{6.5}$$

Or if $[\lim_i H_* X_i]_1 = 0$ we have

$$H_* \operatorname{holim}_i (X_i)_p \cong \lim_i H_* X_i. \tag{6.6}$$

Note that some hypothesis on degree 1 is necessary. Consider the tower

$$K(\mathbb{Z}, 1) \xleftarrow{\times p} K(\mathbb{Z}, 1) \xleftarrow{\times p} K(\mathbb{Z}, 1) \leftarrow \cdots,$$

where $\times p$ means the map multiplies by p on homotopy. Then

$$\operatorname{holim} K(\mathbb{Z}, 1) \simeq K(\mathbb{Z}_p/\mathbb{Z}, 0).$$

However, $\lim_i H_* X_i \cong \mathbb{F}_p$ in degree 0. Note that (6.5) does not apply, and $\operatorname{holim} K(\mathbb{Z}_p, 1) = *$, as predicted by (6.6).

Example 6.8. Let $X: I \to \mathbf{S}$ be an I-diagram of connected simple spaces, again with the property that $H_* X$ is an I-diagram of abelian Hopf algebras. Suppose (for simplicity) that $p > 2$ and that for all $i \in I$, $H_* X(i)$ is concentrated in even degrees. Finally, suppose $\lim^s D_* H_* X = 0$ for $s > 1$, where $D_* H_* X$ is the associated diagram of Dieudonne modules. Write $\lim^1$ for $\lim^1 D_* H_* X$. Then the spectral sequence becomes by Lemma 4.7,

$$R^*_{\mathscr{C}\mathscr{A}} \lim_I H_* X \cong \lim_I H_* X \otimes \Lambda(P \lim^1) \otimes \mathbb{F}_p(R^1 P \lim^1) \Rightarrow H_* \operatorname{holim}_I X_p.$$

The exact sequence

$$0 \to P \lim^1 \to \lim^1 D_* H \xrightarrow{V} \Phi \lim^1 D_* H \to R^1 P \lim^1 \to 0$$

shows $(R^1 P \lim^1)_t = 0$ for $t < 2p$. So if $[\lim^1 D_* H]_2 = 0$ the criterion of 6.4(2) implies that the spectral sequence converges. Also, if the $\lim^s \pi_t X = 0$ for $s > 1$ and $\lim^1 \pi_1 X = 0$, then the computation spectral sequence

$$R^*_{\mathscr{C}\mathscr{A}} \lim_I H_* X \Rightarrow H_* \operatorname{holim}_I X$$

will converge.

In either case, if $R^1 P \lim^1 = 0$, the spectral sequence will collapse because it is a spectral sequence of coalgebras.

If I are the natural numbers (so $X: I \to \mathbf{S}$ is a tower) or I is the category of a finitely generated free group G (so $X: I \to \mathbf{S}$ is a G-space) then $R^1 P \lim^1 = 0$ if V is surjective on $D_* H_* X(i)$ for all $i \in I$. This is because there are exact sequences

$$0 \to \lim_i D_* H_* X \to \prod_i D_* H_* X(i) \to \prod_i D_* H_* X(i) \to \lim_t{}^1 D_* H_* X(i) \to 0$$

in the first case and

$$0 \to H^0(G, D_* H_* X) \to D_* H_* X \to \prod_n D_* H_* X \to H^1(G, D_* H_* X) \to 0$$

in the second.

We close with the following observation. Our homology diagrams are diagrams of Hopf algebras. In practice such may arise from diagrams of loop spaces. Under this assumption, we would like to know whether we have a spectral sequence of Hopf algebras. Note that by [5, Corollary I.7.4] the p-completion of a loop space is a homotopy associative H-space.

Lemma 6.9. *Let $\Omega X : I \to \mathbf{S}$ be a diagram of loop spaces. Then the spectral sequence*

$$R^*_{\mathscr{C}\mathscr{A}} \lim_I H_* \Omega X \Rightarrow H_* \operatorname{holim}(\Omega X)_p$$

is a spectral sequence of Hopf algebras.

Note. The Hopf algebra structure on $R^*_{\mathscr{C}\mathscr{A}} \lim_I H_* \Omega X$ is defined in Example 2.11. If each $X(i)$ is simply connected $(\Omega X(i))_p \simeq \Omega(X(i)_p)$ and

$$\operatorname{holim}(\Omega X)_p \simeq \Omega \operatorname{holim} X_p$$

since holim commutes with loops. The proof will show that the companion spectral sequence of Example 6.6 is also a spectral sequence of Hopf algebras.

Proof. One has a map of I diagrams

$$\Omega X \times \Omega X \to \Omega X$$

induced by H-space multiplication, hence a map of spectral sequences

$$\begin{array}{ccc}
R^*_{\mathscr{C}\mathscr{A}} \lim(H_* \Omega X \otimes H_* \Omega X) & \Rightarrow & H_* \operatorname{holim}(\Omega X \times \Omega X)_p \\
\downarrow & & \downarrow \\
R^*_{\mathscr{C}\mathscr{A}} \lim H_* \Omega X & \Rightarrow & H_* \operatorname{holim}(\Omega X)_p
\end{array}$$

We show that the top spectral sequence is the necessary tensor product spectral sequence. Consider the maps of cosimplicial spaces

$$\begin{aligned}
\operatorname{diag} \Pi^\bullet \mathbb{F}_p^\bullet X \times \operatorname{diag} \Pi \mathbb{F}_p^\bullet X &\leftarrow \Pi^\bullet \Omega X \times \Pi^\bullet \Omega X \\
&\cong \Pi^\bullet(\Omega X \times \Omega X) \to \operatorname{diag} \Pi^\bullet \mathbb{F}_p^\bullet(\Omega X \times \Omega X).
\end{aligned}$$

Then we get a diagram of spectral sequences

$$\begin{array}{ccccc}
R^*_{\mathscr{C}\mathscr{A}} \lim_I H_* \Omega X \otimes R^*_{\mathscr{C}\mathscr{A}} \lim_I H_* \Omega X & \xleftarrow{\cong} & \pi^* H_*(\Pi^\bullet(\Omega X \times \Omega X)) & \xrightarrow{\cong} & R^*_{\mathscr{C}\mathscr{A}} \lim_I (H_* \Omega X \otimes H_* \Omega X) \\
\Downarrow & & \Downarrow & & \Downarrow \\
H_* \operatorname{holim}(\Omega X)_p \otimes H_* \operatorname{holim}(\Omega X)_p & \xleftarrow{f_*} & H_*(\operatorname{holim}(\Omega X \otimes \Omega X)) & \xrightarrow{g_*} & H_* \operatorname{holim}(\Omega X \times \Omega X)_p)
\end{array}$$

Since p-completion and holim commute with finite products, there is a commutative diagram

$$
\begin{array}{ccc}
\operatorname{holim}(\Omega X \times \Omega X) & \xrightarrow{\ f\ } & \operatorname{holim}(\Omega X)_p \times \operatorname{holim}(\Omega X)_p \\
\downarrow g & \nearrow \simeq & \\
\operatorname{holim}(\Omega X \times \Omega X)_p & &
\end{array}
$$

with the diagonal map a homotopy equivalence. The result follows. □

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